

Representing Incomplete Knowledge about Fuzzy Sets

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Abstract

This paper deals with interval-valued fuzzy sets and I-fuzzy sets, Atanassov's intuitionistic fuzzy sets. They are treated in a systematic way as two, formally equivalent, natural extensions of methods of representing incomplete knowledge about sets. We define and investigate triangular norm-based areas of uncertainty of interval-valued fuzzy sets and I-fuzzy sets, and study some properties of related uncertainty degrees of their elements.

Keywords: Interval-valued fuzzy set, I-fuzzy set, Incomplete knowledge, Area of uncertainty.

1 Introduction

In recent years, there is a growing interest in and an intensification of research on two generalizations of the idea of a fuzzy set: interval-valued fuzzy sets and I-fuzzy sets. Both of them are tools for representing incompletely known fuzzy sets, i.e. tools for modeling a mixture of two important phenomena: imprecision and incompleteness of information.

Interval-valued fuzzy sets were introduced by Zadeh in 1975 ([17]). Surprisingly, they were also proposed in the same year by Sambuc [13] and by some other researchers; see [4] for further references. I-fuzzy sets were defined by Atanassov in 1983 under the name "intuitionistic fuzzy sets" ([1, 2]). Since there is a terminological conflict between his objects and "true" intuitionistic fuzzy sets, intuitionistic logic-based fuzzy sets by Takeuti and Titani [14], some alternative names have been proposed, e.g. "bipolar fuzzy sets", "IF-sets", and just "I-fuzzy sets" used in this paper; see [4] for details about that dispute. Speaking formally, interval-valued fuzzy sets and I-fuzzy sets are

equivalent concepts. The idea of I-fuzzy sets, however, puts emphasis on positive and negative information. It seems that just this feature of Atanassov's objects is especially attractive and stimulates their applications in decision making and other areas (see e.g. [10]).

We like to look at interval-valued fuzzy sets and I-fuzzy sets in a systematic way as at two natural extensions of some methods referring to a simpler issue, namely representation of incompletely known sets. For both of these extensions, the notion of a triangular norm-based area of uncertainty and degree of uncertainty will be introduced and investigated.

2 Incomplete Knowledge about Sets and its Representation

Let us take a look at the issue of incomplete knowledge about a set $A \subset M$. The status of some elements $x \in M$ (membership/nonmembership or fulfilment/nonfulfilment of a crisp property, in other words) is then practically unknown or uncertain. This implies a division of the universe M into the following three pairwise disjoint classes:

A^+ - composed of those x 's whose belonging to A is known (sure),

A^- - contains the x 's that surely do not belong to A ,

$A^?$ - the *area of uncertainty* of A , composed of all x 's with unknown (uncertain) status.

Since each of these classes is uniquely determined by the remaining two, incompleteness of knowledge about A can be modeled by specifying any of the pairs

$$(A^+, A^?), (A^-, A^?), (A^+, A^-)$$

of sets.

If the representation $(A^+, A^?)$ is chosen, it is convenient to rewrite it in the form $(A^+, A^+ \cup A^?)$. So, speaking

more formally, incompleteness of knowledge about A can be expressed by means of a pair

$$(B, C) \text{ with } B \subset C, \quad (2.1)$$

where the set B contains sure elements of A , and C represents sure elements together with those with unknown or uncertain status. Then

$$A^? = B' \cap C. \quad (2.2)$$

Recollect in the context of (2.1) that each nondecreasing, uncountable family of (cut) sets

$$(A_{1-t})_{t \in [0,1]}$$

is identical with a fuzzy set $A: M \rightarrow [0,1]$ representing imprecise information, where $A_w = \{x: A(x) \geq w\}$. The two-element families from (2.1) are isomorphic to fuzzy sets $M \rightarrow \{0, 1/2, 1\}$.

Pairs (2.1) were introduced by Gentilhomme in 1968 as *flo sets* ([6]; see also [9]). Again, independently and using different motivations, they were defined at the same time by Klaua as *partial sets* ([7]). Pawlak's *rough sets* from [11, 12] are also constructions collapsing in essence to (2.1). However, contrary to flo sets, their sums and intersections cannot be constructed by performing suitable set-theoretic operations on the components. Recall that a rough set

$$\mathfrak{R}_A = (\underline{A}, \overline{A}) \quad (2.3)$$

is a pair of approximating sets in which

$$\underline{A} = \{x \in M: [x]_\rho \subset A\}$$

is the *lower approximation* of A , whereas

$$\overline{A} = \{x \in M: [x]_\rho \cap A \neq \emptyset\}$$

forms its *upper approximation*. The equivalence relation ρ is treated as an indiscernibility relation of objects x described in terms of given attributes and their values. The area of uncertainty (2.2),

$$A' \cap \overline{A},$$

is then called the *boundary* of A with respect to ρ .

An interesting generalization of the representation from (2.1) is the concept of a *twofold fuzzy set*

$$\mathbb{T}_A = (S_A, P_A) \text{ with } S_A \subset (P_A)_1 \quad (2.4)$$

introduced in [5]. S_A is a fuzzy set of more or less sure elements of A , whereas P_A forms a fuzzy set of its more

or less possible elements. The inclusion in (2.4) implies $S_A \subset P_A$, and means that elements being sure to a positive degree must be fully possible. The corresponding area of uncertainty

$$S_A' \cap P_A$$

is interpreted as a fuzzy set of *dubious elements* of A , i.e. elements whose belonging to A is possible but not sure.

Let us move on to the other two variants of representations of an incompletely known set A . It seems that the form $(A^-, A^?)$, dual in a way to $(A^+, A^?)$ and focusing on negative information, is not used in the subject literature. Formally, it can be rewritten as (2.1), too, but the semantics of the components is then different: B is a set of impossible elements of A , whereas C contains elements which are impossible or uncertain.

Much more interesting seems to be the third form, namely (A^+, A^-) . It is bipolar by putting emphasis on positive (A^+) and negative (A^-) information, on positive and negative examples of elements of A , in other words. This way of representing incomplete knowledge is exemplified by the idea of *sub-definite sets* by Narin'yani ([8]). In a more formalized notation, incomplete knowledge about A is now represented by a pair

$$(B, D) \text{ with } B \subset D'. \quad (2.5)$$

B is a set of sure elements of A , and D contains sure elements of A' . We thus have

$$B \subset A \subset D', \quad (2.6)$$

i.e. B and D' , respectively, are a lower evaluation and an upper evaluation of A . The set

$$A^? = B' \cap D' \quad (2.7)$$

forms the area of uncertainty of A .

3 Representing Incompletely Known Fuzzy Sets

Trying to model fuzzy sets with incompletely known membership grades, a reasonable and natural way of doing is to pattern oneself upon the representations of incompletely known sets from Section 2 with special reference to (2.1) and (2.5). And this idea finds its materialization in the concepts of interval-valued fuzzy sets and I-fuzzy sets (cf. also [15]). Let us look at them a bit closer drawing special attention to related areas of uncertainty.

3.1 Using Interval-Valued Fuzzy Sets

As one knows, an interval-valued fuzzy set is a pair $E = (A_l, A_u)$ of fuzzy sets $A_l, A_u: \mathbf{M} \rightarrow [0, 1]$ with

$$A_l \subset A_u.$$

A_l and A_u , respectively, form a lower bound and an upper bound of an incompletely known fuzzy set A , i.e. $A(x) \in [A_l(x), A_u(x)]$. What we deal with is thus a direct extension of (2.1) to fuzzy sets.

For a triangular norm t and a strong negation v , let us define (cf. (2.2))

$$U_E = A_l^v \cap_t A_u. \quad (3.1)$$

The fuzzy set U_E is the area of uncertainty of E , where as

$$U_E(x) = v(A_l(x)) t A_u(x) \quad (3.2)$$

is a *degree of uncertainty* concerned with x . Worth considering are properties of (3.2) for two important classes of t-norms. To this end, however, we need some additional notation. Denote by v_t the negation induced by t , $v_t(a) = \bigvee \{c: a t c = 0\}$. On the other hand, t^v will denote the t-conorm which is v -dual to t , $a t^v b = v(v(a) t v(b))$.

If t is nonstrict Archimedean and $v = v_t$, then $U_E(x)$ expresses our uncertainty as to membership degree of x in an incompletely known fuzzy set A represented by E . Notice that

$$U_E(x) = 0 \Leftrightarrow A_l(x) = A_u(x), \quad (3.3)$$

i.e.

$$U_E = 1_{\emptyset} \Leftrightarrow E \text{ is a fuzzy set.}$$

Interpreting (3.2) becomes a more complex task whenever we use a strong negation v and a t-norm t having no zero divisors, say, a strict t or $t = \wedge$. This time $U_E(x)$ seems to be uncertainty as to membership degree of x in A combined with fuzziness of that degree. Indeed, we have

$$U_E(x) = 0 \Leftrightarrow (A_l(x), A_u(x)) \in \{(0, 0), (1, 1)\}, \quad (3.4)$$

i.e.

$$U_E = 1_{\emptyset} \Leftrightarrow E \text{ is a set.}$$

If there is no uncertainty as to membership degrees in A , then E collapses to a fuzzy set, $E = (A, A)$, and

$$U_E(x) = v(A(x)) t A(x). \quad (3.5)$$

So, $U_E(x)$ becomes an ordinary t -based index of fuzziness of x .

Theorem 3.1. *Let $E = (A_l, A_u)$. If t is a nonstrict Archimedean t-norm, $v = v_t$ and h is normed generator of t^v , then*

$$\forall x \in \mathbf{M}: U_E(x) = h^{-1}(h(A_u(x)) - h(A_l(x))).$$

In the case of nonstrict Archimedean t-norms, uncertainty degrees $U_E(x)$ are thus isomorphic to the lengths of the intervals $[A_l(x), A_u(x)]$. Denote by $v_{\mathbb{L}}$ and $t_{\mathbb{L}}$, respectively, the Łukasiewicz negation and the Łukasiewicz triangular norm, respectively, where $v_{\mathbb{L}}(a) = 1 - a$ and $a t_{\mathbb{L}} b = 0 \vee (a + b - 1)$. Applying $t = t_{\mathbb{L}}$, we get $v_t = v_{\mathbb{L}}$ and $h = id$, which gives

$$U_E(x) = A_u(x) - A_l(x).$$

More generally, one has

$$U_E(x) = ((A_u(x))^p - (A_l(x))^p)^{1/p}$$

when a Schweizer-Sklar t-norm $t_{S,p}$ is used, where

$$a t_{S,p} b = [0 \vee (a^p + b^p - 1)]^{1/p}, \quad p > 0.$$

Theorem 3.2. *Let $E = (A_l, A_u)$ be an interval-valued fuzzy set. One has*

$$\forall x \in \mathbf{M}: A_l(x) t^v U_E(x) = A_u(x)$$

if t is nonstrict Archimedean and $v = v_t$. For $t = \wedge$ with any strong negation v , we get

$$\forall x \in \mathbf{M}: A_l(x) t^v U_E(x) \leq A_u(x).$$

For $t = t_{\mathbb{L}}$, the equality in Theorem 3.2 collapses to the standard

$$A_l(x) + U_E(x) = A_u(x).$$

There is no general relationship between $A_l(x) t^v U_E(x)$ and $A_u(x)$ and if a strict t is used.

3.2 Using I-Fuzzy Sets

By an *I-fuzzy set* one means a pair $\mathcal{E} = (A^+, A^-)$ of fuzzy sets $A^+, A^-: \mathbf{M} \rightarrow [0, 1]$ such that

$$A^+ \subset (A^-)^v, \quad (3.6)$$

where v denotes a strong negation. This notion is thus a direct extension of (2.5). By virtue of (3.6), each I-fuzzy set is formally equivalent to an interval-valued fuzzy set. However, some care as to that equivalence is necessary if triangular norm-based operations are taken into account. Basic relations for and operations on I-fuzzy

sets $\mathcal{E} = (A^+, A^-)$ and $\mathcal{F} = (B^+, B^-)$ with (3.6) are then defined as follows:

$$\begin{aligned} \mathcal{E} = \mathcal{F} &\Leftrightarrow A^+ = B^+ \ \& \ A^- = B^-, \\ \mathcal{E} \subset \mathcal{F} &\Leftrightarrow A^+ \subset B^+ \ \& \ B^- \subset A^-, \\ \mathcal{E} \cup_{t,s} \mathcal{F} &= (A^+ \cup_s B^+, A^- \cap_t B^-), \\ \mathcal{E} \cap_{t,s} \mathcal{F} &= (A^+ \cap_t B^+, A^- \cup_s B^-), \\ \mathcal{E}' &= (A^-, A^+), \end{aligned} \quad (3.7)$$

where t and s , resp., denote a t-norm and a t-conorm, $(F \cap_t G)(x) = F(x) t G(x)$, $(F \cup_s G)(x) = F(x) s G(x)$. By the way, t and s can be, respectively, replaced by an interval-valued (non-representable) t-norm and t-conorm from [3]. There are good reasons for taking $s = t^v$ in (3.7) (see [10, 16] for details). Indeed, just the use of $s = t^v$ guarantees that sums and intersections of I-fuzzy sets coincide with s -based sums and t -based intersections of interval-valued fuzzy sets corresponding to them, where

$$\begin{aligned} (A_l, A_u) \cup_s (B_l, B_u) &= (A_l \cup_s B_l, A_u \cup_s B_u) \\ \text{and} \\ (A_l, A_u) \cap_t (B_l, B_u) &= (A_l \cap_t B_l, A_u \cap_t B_u). \end{aligned}$$

What is more, sums and intersections of I-fuzzy sets collapsing to ordinary fuzzy sets are then (and only then) still fuzzy sets expressed as I-fuzzy sets. Notice that, on the other hand, the very idea of interval-valued fuzzy sets with s -based sums and t -based intersections does not require the restriction $s = t^v$ in this context.

Unlike interval-valued fuzzy sets, the idea of I-fuzzy sets puts emphasis on positive (A^+) and negative (A^-) information, on positive and negative examples of elements having an imprecise property. $\mathcal{E} = (A^+, A^-)$ represents incomplete knowledge about a fuzzy set A and its complement A^v which are such that (cf. (2.6))

$$A^+ \subset A \subset (A^-)^v \quad \text{and, hence,} \quad A^- \subset A^v \subset (A^+)^v.$$

So, A^+ is a lower bound on A , A^- forms a lower bound on A^v ,

$$A(x) \in [A^+(x), v(A^-(x))]$$

and

$$A^v(x) \in [A^-(x), v(A^+(x))].$$

In the original commonly used Atanassov's terminology from [1, 2], A^+ is understood as a *membership function*, whereas A^- is interpreted as a *nonmembership function*. So, $A^+(x)$ forms a *membership degree* of x , and $A^-(x)$ is a *nonmembership degree*. Condition (3.6) guarantees that

$$A^+(x) \leq a^* \quad \text{or} \quad A^-(x) \leq a^*$$

for each x with a^* denoting a unique fixed point of the negation v . In other words, the membership and the

nonmembership degrees of x must not be "too large" at the same time.

Drawing inspiration from Pawlak's rough set theory, the following two types of approximate equalities can be defined on I-fuzzy sets:

$$\mathcal{E} =_{pos} \mathcal{F} \Leftrightarrow A^+ = B^+, \quad (\mathcal{E} \text{ positively equal to } \mathcal{F})$$

$$\mathcal{E} =_{neg} \mathcal{F} \Leftrightarrow A^- = B^-. \quad (\mathcal{E} \text{ negatively equal to } \mathcal{F})$$

Both $=_{pos}$ and $=_{neg}$ are obviously equivalence relations, and we have

$$\mathcal{E} = \mathcal{F} \Leftrightarrow \mathcal{E} =_{pos} \mathcal{F} \ \& \ \mathcal{E} =_{neg} \mathcal{F}.$$

Similarly, approximate inclusions

$$\mathcal{E} \subset_{pos} \mathcal{F} \Leftrightarrow A^+ \subset B^+ \quad (\mathcal{E} \text{ positively contained in } \mathcal{F})$$

and

$$\mathcal{E} \subset_{neg} \mathcal{F} \Leftrightarrow B^- \subset A^- \quad (\mathcal{E} \text{ negatively contained in } \mathcal{F})$$

can be introduced as two partial order relations.

3.3 Uncertainty Areas of I-Fuzzy Sets

The uncertainty area of an I-fuzzy set $\mathcal{E} = (A^+, A^-)$ is usually called the *hesitation area* of \mathcal{E} , and is defined as ([10, 16])

$$\chi_{\mathcal{E}} = (A^+)^v \cap_t (A^-)^v = (A^+ \cup_{t^v} A^-)^v \quad (3.8)$$

with a t-norm t and a strong negation v (see (2.7)). Its membership degree

$$\chi_{\mathcal{E}}(x) = v(A^+(x) t^v A^-(x)) \quad (3.9)$$

is interpreted as a *degree of hesitation* concerned with x . Notice that

$$\forall x \in \mathbf{M}: \phi(A^+(x) t^v A^-(x)) + \phi(\chi_{\mathcal{E}}(x)) = 1 \quad (3.10)$$

with ϕ being an automorphism of $[0, 1]$ generating v , $v(a) = \phi^{-1}(1 - \phi(a))$.

The original Atanassov's formulation of I-fuzzy sets from [1, 2] is a particular case of the presented general approach when $\cup_{\wedge, v}$ and $\cap_{\wedge, v}$ are used in (3.7) as basic operations, whereas $t = t_{\wedge}$ and $v = v_{\wedge}$ are employed in (3.6) and (3.8). Indeed, (3.6), (3.9) and (3.10), respectively, then collapse to (see also Theorem 3.3)

$$\begin{aligned}
A^+(x) + A^-(x) &\leq 1, \\
\chi_{\mathcal{E}}(x) &= 1 - A^+(x) - A^-(x), \\
A^+(x) + A^-(x) + \chi_{\mathcal{E}}(x) &= 1
\end{aligned}$$

for each $x \in \mathbf{M}$.

Let us restrict our further discussion to Archimedean t-norms, including $\mathbf{t} = \wedge$. First, we present a few additional examples of hesitation degrees. For $\mathbf{t} = \mathbf{t}_{S,p}$ with $\mathbf{v} = \mathbf{v}_p$, the basic condition (3.6) collapses to

$$\forall x \in \mathbf{M}: (A^+(x))^p + (A^-(x))^p \leq 1,$$

and we obtain

$$\chi_{\mathcal{E}}(x) = [1 - (A^+(x))^p - (A^-(x))^p]^{1/p}.$$

Using the product t-norm with $\mathbf{v} = \mathbf{v}_{\mathbf{L}}$, one gets

$$\chi_{\mathcal{E}}(x) = 1 - A^+(x) - A^-(x) + A^+(x) \cdot A^-(x),$$

whereas $\mathbf{t} = \wedge$ gives

$$\chi_{\mathcal{E}}(x) = 1 - A^+(x) \vee A^-(x).$$

Similarly to (3.3), if \mathbf{t} is nonstrict Archimedean and $\mathbf{v} = \mathbf{v}_p$, then $\chi_{\mathcal{E}}(x)$ expresses our hesitation as to membership degree of x in an incompletely known fuzzy set A modeled by \mathcal{E} . One has

$$\chi_{\mathcal{E}}(x) = 0 \Leftrightarrow A^+(x) = \mathbf{v}(A^-(x))$$

for each x , which leads to

$$\chi_{\mathcal{E}} = 1_{\emptyset} \Leftrightarrow \mathcal{E} \text{ is a fuzzy set.}$$

As a corollary from Theorem 3.1, we have

Theorem 3.3. *Let $\mathcal{E} = (A^+, A^-)$. If \mathbf{t} is nonstrict Archimedean, $\mathbf{v} = \mathbf{v}_p$, and h is normed generator of \mathbf{t}^{\vee} , then*

$$\forall x \in \mathbf{M}: \chi_{\mathcal{E}}(x) = h^{-1}(1 - h(A^+(x)) - h(A^-(x))).$$

Again, applying a strong negation \mathbf{v} with a strict t-norm \mathbf{t} or $\mathbf{t} = \wedge$, $\chi_{\mathcal{E}}(x)$ becomes hesitation as to membership degree of x combined with fuzziness of that degree, namely (cf. (3.4), (3.5))

$$\chi_{\mathcal{E}}(x) = 0 \Leftrightarrow (A^+(x), A^-(x)) \in \{(1, 0), (0, 1)\},$$

i.e.

$$\chi_{\mathcal{E}} = 1_{\emptyset} \Leftrightarrow \mathcal{E} \text{ is a set.}$$

Further, if \mathcal{E} collapses to a fuzzy set, $\mathcal{E} = (A, A^{\vee})$, and thus there is no hesitation as to membership degrees in A , we have

$$\chi_{\mathcal{E}}(x) = A(x) \mathbf{t} \mathbf{v}(A(x)),$$

i.e. $\chi_{\mathcal{E}}(x)$ forms an ordinary \mathbf{t} -based index of fuzziness of x .

Theorem 3.4. *Let $\mathcal{E} = (A^+, A^-)$. If $\mathbf{v} = \mathbf{v}_t$ with a nonstrict Archimedean t-norm \mathbf{t} , then*

$$A^+ \cup_{\mathbf{t}^{\vee}} \chi_{\mathcal{E}} = (A^-)^{\vee}.$$

For $\mathbf{t} = \wedge$ with any strong negation \mathbf{v} ,

$$A^+ \cup_{\mathbf{t}^{\vee}} \chi_{\mathcal{E}} \subset (A^-)^{\vee}.$$

This analogue of Theorem 3.2 refers to the identity $A^+ \cup A^{\vee} = (A^-)^{\vee}$ for incompletely known sets. As previously, if a strict t-norm is used, there is no general relationship between $A^+ \cup_{\mathbf{t}^{\vee}} \chi_{\mathcal{E}}$ and $(A^-)^{\vee}$.

Finally, we like to present an illustrative example of applications of triangular norm-based I-fuzzy sets and their hesitation areas in GDM, group decision making. Let $\mathbf{P} = \{p_1, \dots, p_m\}$ be a set of $m \geq 1$ individuals, and let $\mathbf{S} = \{s_1, \dots, s_n\}$ denote a set of $n \geq 2$ options (decision alternatives). Each individual p_k , $k = 1, \dots, m$, formulates his/her individual preferences over \mathbf{S} by means of a binary fuzzy relation R_k in \mathbf{S} . R_k can be presented in the form of an $n \times n$ matrix

$$R_k = [r_{ij}^k]$$

in which $r_{ij}^k \in [0, 1]$ is a degree to which p_k prefers s_i to s_j ; one puts $r_{ii}^k = 0$. Intuitively speaking, r_{ij}^k and r_{ji}^k cannot be "too large" at the same time and, thus, we assume that

$$r_{ij}^k \leq \mathbf{v}(r_{ji}^k)$$

with a strong negation \mathbf{v} (cf. (3.6)). The number

$$h_{ij}^k = \mathbf{v}(r_{ij}^k) \mathbf{t} \mathbf{v}(r_{ji}^k)$$

with $i \neq j$ and a t-norm \mathbf{t} expresses a hesitation of p_k as to his/her preference concerning s_i and s_j (cf. (3.9)); additionally, one defines $h_{ii}^k = 0$. This formal model is then a starting point for constructing a family of general algorithms of GDM, including the direct and indirect approach. In the direct case, they generate a solution being a fuzzy set S_Q of options such that a soft majority, Q individuals, is not against them, where Q denotes a "most"-type relative linguistic quantifier. Using the indirect approach, first the individual preference matrices R_1, \dots, R_m are aggregated into a social preference $n \times n$ matrix R representing common preferences of the whole group \mathbf{P} over \mathbf{S} . Second, on the basis of R , one derives a solution W_Q understood in this case as a fuzzy set of

options being preferred by the group over Q other options from S . The reader is referred to [10] for a detailed presentation and discussion of this issue.

4 Concluding Remarks

In this paper, first, we have dealt with the issue of sets with incomplete knowledge for which the status of some elements is practically unknown or uncertain. Three mathematically equivalent methods of representing that incompleteness and related uncertainty are then possible. Further, we have considered interval-valued fuzzy sets and I-fuzzy sets, Atanassov's intuitionistic fuzzy sets, with triangular norms. Those objects were treated as two formally equivalent tools for representing incomplete knowledge about fuzzy sets which are natural extensions of the methods of representing incompletely known sets. We have defined and investigated triangular norm-based uncertainty areas of interval-valued fuzzy sets and I-fuzzy sets. In general, their properties and semantics depend on the t-norm being used. In the case of a t-norm without zero divisors, an interpretation involving fuzziness degrees seems to be appropriate.

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