

On Additive and Multiplicative Fuzzy Models*

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Abstract

Systems which use a fuzzy rule base and an inference mechanisms are quite frequently used in many applications. Fuzzy rules and inference mechanisms can be described by a system of fuzzy relation equations. A solution to a given system of fuzzy relation equations can serve us a proper model of fuzzy rules (fuzzy model for short). But only two particular solutions, let us call them disjunctive and conjunctive fuzzy models, have been mainly studied so far.

The paper aims at new solutions of systems of fuzzy relation equations. The first one - additive fuzzy model - which is derived from the disjunctive fuzzy model was motivated by several methods often used in practice. The second one — multiplicative fuzzy model — which is derived from the conjunctive fuzzy model was neglected up to now. The paper investigates both new solutions and describes existence of a direct connection between systems of fuzzy relation equations and different fuzzy models. The crucial role of the Ruspini partition is proved and also visually demonstrated.

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1 Introduction

Fuzzy rule based (FRB) systems have been many times

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demonstrated to be powerful tools in modelling, decision making or automatic control. They basically consist of two main blocks: a fuzzy rule base determined by a set of n fuzzy rules and an inference mechanism. A correct choice of a fuzzy relation which models a fuzzy rule base and a corresponding inference mechanism is crucial for a proper performance of the whole FRB.

Consider arbitrary universes X and Y . The classes of fuzzy sets on X and Y are denoted by $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, respectively. Then all information available in a given fuzzy rule base is contained in pairs of input-output fuzzy sets i.e. in $(\mathbf{A}_1, \mathbf{B}_1), \dots, (\mathbf{A}_n, \mathbf{B}_n)$, telling us that, for $i = 1, \dots, n$, the fuzzy set $\mathbf{A}_i \in \mathcal{F}(X)$ is assigned the fuzzy set $\mathbf{B}_i \in \mathcal{F}(Y)$ [31].

In general, there are two standard approaches to modelling a given fuzzy rule base by an appropriate fuzzy relation. Let $x \in X$, $y \in Y$, let operation $*$ be a left-continuous t-norm and finally, let \rightarrow_* be its adjoint residuation operation [18]. Then the first approach consists in a construction of a fuzzy relation $\hat{\mathbf{R}}_* \in \mathcal{F}(X \times Y)$ given as follows

$$\hat{\mathbf{R}}_*(x, y) = \bigwedge_{i=1}^n (\mathbf{A}_i(x) \rightarrow_* \mathbf{B}_i(y)). \quad (1)$$

As written in [11], “In the above view, each piece of information (fuzzy rule) is viewed as a constraint. This view naturally leads to a conjunctive way of merging the individual pieces of information since the more information, the more constraints and the less possible values to satisfy them.”

This fact together with the fact that the minimum operation as well as other t-norms are appropriate interpretations of a conjunction (logical connective AND) and residuation operations are appropriate interpretations of an implication [1, 10, 15, 23, 30] leads to a conclusion that $\hat{\mathbf{R}}_*$ is a proper model of the following

fuzzy rules

$$\begin{array}{l}
\text{IF } x \text{ is } \mathcal{A}_1 \quad \text{THEN} \quad y \text{ is } \mathcal{B}_1 \\
\quad \quad \quad \dots \\
\quad \quad \quad \text{AND} \\
\quad \quad \quad \dots \\
\text{IF } x \text{ is } \mathcal{A}_n \quad \text{THEN} \quad y \text{ is } \mathcal{B}_n
\end{array} \quad (2)$$

where $\mathcal{A}_i, \mathcal{B}_i$ are membership predicates represented by fuzzy sets $\mathbf{A}_i \in \mathcal{F}(X)$ and $\mathbf{B}_i \in \mathcal{F}(Y)$, respectively.

The second standard approach to modelling a fuzzy rule base, initiated by [20], consists in a construction of a fuzzy relation $\check{\mathbf{R}}_* \in \mathcal{F}(X \times Y)$ given as follows

$$\check{\mathbf{R}}_*(x, y) = \bigvee_{i=1}^n (\mathbf{A}_i(x) * \mathbf{B}_i(y)). \quad (3)$$

Obviously, fuzzy relation (3) can be hardly considered as a model of fuzzy rule base (2).

We again recall the work of Dubois, Prade and Ughetto [11]. "It seems that fuzzy rules modelled by (3) are not viewed as constraints but are considered as pieces of data. Then the maximum in (3) expresses accumulation of data".

This fact together with a commonly known fact that the maximum operation as well as other t-conorms are appropriate interpretations of a disjunction (logical connective OR) [15, 23] leads to a conclusion that $\check{\mathbf{R}}_*$ given by (3) is a proper model of the following set of fuzzy rules

$$\begin{array}{l}
x \text{ is } \mathcal{A}_1 \quad \text{AND} \quad y \text{ is } \mathcal{B}_1 \\
\quad \quad \quad \dots \\
\quad \quad \quad \text{OR} \\
\quad \quad \quad \dots \\
x \text{ is } \mathcal{A}_n \quad \text{AND} \quad y \text{ is } \mathcal{B}_n.
\end{array} \quad (4)$$

It is worth mentioning that distinguishing between the conditional (IF-THEN) form of fuzzy rules (2) and the Cartesian product (AND) form of fuzzy rules (4) on a syntactical level is not very common but it can be found e.g. in [3, 17, 22]. Usually only the form given by (2) is considered, because of several, e.g., historical reasons and the differences are taken into account only on the semantical level. But the differences can play a crucial role for further implementations and, therefore, they should be kept in mind. For a more detailed study concerning both rule forms we refer to [15, 23, 17] and to an exhaustive investigation in [9] and in [22].

2 Systems of Fuzzy Relation Equations

Besides a fuzzy rule base, (fuzzy) inference mechanism is an essential part of each FRB system. It is a deduction rule determining an output $\mathbf{B} \in \mathcal{F}(Y)$ based on an

arbitrary input $\mathbf{A} \in \mathcal{F}(X)$. In particular, it is defined as an image of \mathbf{A} under a fuzzy relation $\mathbf{R} \in \mathcal{F}(X \times Y)$, which models a given fuzzy rule base. In most cases, we deal with the *direct image* (sup-* composition)

$$\mathbf{B} = \mathbf{A} \circ_* \mathbf{R} \quad (5)$$

which stems from the *compositional rule of inference* introduced by L.A. Zadeh [32]. It is given by

$$(\mathbf{A} \circ_* \mathbf{R})(y) = \bigvee_{x \in X} (\mathbf{A}(x) * \mathbf{R}(x, y)) \quad (6)$$

and it is worth mentioning that its logical background coincides with the *generalized modus ponens* [15].

A fuzzy rule base may be viewed as a partial mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$. Building a fuzzy inference module on the base of a rule base means extending this partial function to a total one in some "reasonable manner". It means we have to associate $\mathbf{B} \in \mathcal{F}(Y)$ with an arbitrary $\mathbf{A} \in \mathcal{F}(X)$ in such a way which having an input \mathbf{A}_i would determine an output equal to \mathbf{B}_i for $i = 1, \dots, n$.

It leads to the following *system of direct image fuzzy relation equations*

$$\mathbf{A}_i \circ_* \mathbf{R} = \mathbf{B}_i \quad i = 1, \dots, n. \quad (7)$$

A fuzzy relation $\mathbf{F} \in \mathcal{F}(X \times Y)$ which fulfills the equality (7) is a *solution* to the system of direct image equations.

Let us recall some basic results concerning the systems of direct image equations which can be found e.g. in [6, 16, 19].

Theorem 1 *System (7) is solvable if and only if $\hat{\mathbf{R}}_*$ is a solution to the system and moreover, by $\hat{\mathbf{R}}_*$ is the greatest solution to (7).*

Theorem 1 is a crucial theorem in the field of direct image equations since it is a necessary and sufficient condition of the solvability of system (7) and it determines a solution. Moreover, whenever we deal with fuzzy rules (2) interpreted by $\hat{\mathbf{R}}_*$, the direct image is the first choice for an inference mechanism since fuzzy relation $\hat{\mathbf{R}}_*$ has a unique position in the set of all possible solutions. There cannot appear any other solutions if $\hat{\mathbf{R}}_*$ is not a solution and it is the greatest solution if the corresponding system is solvable.

Let us recall a solvability criterion (sufficient condition) published in [7] and then independently re-found in [16] specifying conditions upon which even $\check{\mathbf{R}}_*$ can be a solution to system (7).

Theorem 2 *Let \mathbf{A}_i for $i = 1, \dots, n$ be normal. Then $\check{\mathbf{R}}_*$ is a solution to (7) if and only if the following*

condition

$$\bigvee_{x \in X} (\mathbf{A}_i(x) * \mathbf{A}_j(x)) \leq \bigwedge_{y \in Y} (\mathbf{B}_i(y) \leftrightarrow_* \mathbf{B}_j(y)) \quad (8)$$

holds for arbitrary $i, j \in \{1, \dots, n\}$.

Theorem 2 specifies a condition under which $\check{\mathbf{R}}_*$, connected to the direct image inference mechanism, is an appropriate model of a fuzzy rule base. On the other hand, whenever $\check{\mathbf{R}}_*$ is an appropriate model, the fuzzy relation $\hat{\mathbf{R}}_*$ is an appropriate model too. So, $\check{\mathbf{R}}_*$ does not achieve such a unique position among other possible fuzzy models compared to $\hat{\mathbf{R}}_*$.

But it is not common approach to fix an inference mechanism and search for an appropriate model of a given fuzzy rule base which would be a solution to a corresponding system of fuzzy relation equations. Vice-versa, usually a fuzzy relation is chosen first based on certain, here more closely unspecified, reasons. Therefore, it is natural to adopt another inference mechanism which yields a system of fuzzy relation equations where $\check{\mathbf{R}}_*$ has an analogous position among other solutions to the system.

Let us recall by subdirect image (inf- \rightarrow_* composition)

$$\mathbf{B} = \mathbf{A} \triangleleft_* \mathbf{R} \quad (9)$$

which rises from the *Bandler-Kohout subproduct* [2]. It is given by

$$(\mathbf{A} \triangleleft_* \mathbf{R})(y) = \bigwedge_{x \in X} (\mathbf{A}(x) \rightarrow_* \mathbf{R}(x, y)). \quad (10)$$

The subdirect image has no connection to the generalized modus ponens deductive rule and its motivation was different in [2]. On the other hand, as mentioned in [14], the inference mechanism needs not be necessarily logical but simply a mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ fulfilling certain properties.

In the sequel, we will recall some basic facts [6, 19] about *systems of subdirect image fuzzy relation equations* given as follows

$$\mathbf{A}_i \triangleleft_* \mathbf{R} = \mathbf{B}_i \quad i = 1, \dots, n \quad (11)$$

which should justify our further usage of the subdirect image as an inference mechanism.

Theorem 3 *System (11) is solvable if and only if $\check{\mathbf{R}}_*$ is a solution to the system and moreover, $\check{\mathbf{R}}_*$ is the least solution to system (11).*

Furthermore, both systems of fuzzy relation equations are dual [6].

Theorem 4 [21] *Let \mathbf{A}_i for $i = 1, \dots, n$ be normal. Then $\hat{\mathbf{R}}_*$ is a solution to (11) if and only if the condition*

$$\bigvee_{x \in X} (\mathbf{A}_i(x) * \mathbf{A}_j(x)) \leq \bigwedge_{y \in Y} (\mathbf{B}_i(y) \leftrightarrow_* \mathbf{B}_j(y))$$

holds for arbitrary $i, j \in \{1, \dots, n\}$.

Here we observe, that $\check{\mathbf{R}}_*$ has precisely the same position among other solutions to system (11) as the fuzzy relation $\hat{\mathbf{R}}_*$ had in the case of the direct image equations. So, if we adopt the idea of [14] that the inference mechanism can be understood only as a certain mapping between collections of fuzzy sets without deeper connection to a logical deduction rule, nothing prevents us from treating the subdirect image as a specific kind of finding a conclusion behaving as an inference mechanism.

Based on the facts and theorems above, we claim that fuzzy rules (2) predetermine the direct image inference while fuzzy rules (4) predetermine the subdirect image inference.

Remark 1 *It is worth mentioning that condition (8) appearing in Theorem 2 and Theorem 4 is from the practical point of view not very convenient. On the other hand, it is well known fact that if antecedent fuzzy sets form the so called $*$ semi-partition [8] it forces fulfilling the discussed condition in advance.*

3 Additive Models

A lot of work has been done in the field of fuzzy relation equations [6, 13, 16], mainly aiming at the solvability criteria, greatest and lower solutions, least and greater solutions, respectively and at determining complete solution sets.

Unfortunately, this work has not been followed by practitioners very often. This was perhaps caused, besides other unspecified reasons, by the fact that neuro-fuzzy systems [12] and Takagi-Sugeno rules [29] are the most frequently used fuzzy methods nowadays. But only two particular solutions $\hat{\mathbf{R}}_*$ and $\check{\mathbf{R}}_*$ have been mainly studied so far [24].

Let us only briefly recall that Takagi and Sugeno proposed fuzzy rules of the form

$$\mathbf{IF} \ x \text{ is } \mathcal{A}_i \quad \mathbf{THEN} \quad y \text{ is } f_i(x), \quad i = 1, \dots, n \quad (12)$$

which are modelled by the weighted arithmetic mean

$$y = \frac{\sum_{i=1}^n \mathbf{A}_i(x) f_i(x)}{\sum_{i=1}^n \mathbf{A}_i(x)}, \quad (13)$$

which has obviously nothing common with logical implication and so, they do not correspond to the fuzzy

rules in conditional form (2). But because of powerful approximation abilities, they became very popular in fuzzy community.

Let us stress that the consequent parts $f_i(x)$ of rules (12) are usually polynomial functions so, we can distinguish Takagi-Sugeno rules of the k -th order where $k \in \mathbb{N}$ denotes the order of the consequent polynomial functions.

It is worth mentioning that the so called *Ruspini* condition [28] given by

$$\sum_{i=1}^n \mathbf{A}_i(x) = 1, \quad \text{for } x \in X \quad (14)$$

is often required and therefore, the interpretation of the Takagi-Sugeno rules of the 0-th order is then given by the formula

$$\sum_{i=1}^n \mathbf{A}_i(x) b_i \quad (15)$$

where $b_i \in \mathbb{R}$ are consequents of rules (12).

Remark 2 *Takagi-Sugeno rules are basically data-driven i.e. determined on the basis of a finite input-output data set. Besides this standard approach to an identification of a model, there exists an integral (continuous) version of the identification of the Takagi-Sugeno rules of the 0-th order called fuzzy transform [25].*

Indeed, considering crisp values b_i to be singletons, i.e., special fuzzy numbers on Y , having in mind that the product is nothing else but a particular t-norm [18], and the fact that we assume the Ruspini condition leads to the following natural generalization of Takagi-Sugeno rules with fuzzy consequents

$$\mathbf{R}_*^\oplus(x, y) = \bigoplus_{i=1}^n (\mathbf{A}_i(x) * \mathbf{B}_i(y)), \quad (16)$$

where \oplus is the Łukasiewicz t-conorm. Moreover, it coincides with standard fuzzy relations appearing in the neuro-fuzzy systems [12].

Fuzzy relation $\mathbf{R}_*^\oplus \in \mathcal{F}(X \times Y)$ is nothing else but a special case of a model of fuzzy rule base (4) derived from the disjunctive model $\tilde{\mathbf{R}}_*$ where the disjunction aggregating particular pieces of data is generally given by a t-conorm, in this particular case by the Łukasiewicz addition. Therefore, it will be called *additive* fuzzy model.

Remark 3 *Formalization of fuzzy rules (2) modelled by $\tilde{\mathbf{R}}_*$ can be viewed in the “conjunctive normal forms” (CNF) and formalization of fuzzy rules (4) modelled by \mathbf{R}_* can be viewed in the “disjunctive normal forms”*

(DNF), for both we refer to [26]. The normal forms were proposed to investigate fuzzy models from an approximation point of view. The additive models \mathbf{R}_^\oplus correspond to the “additive normal forms” (ANF) which were motivated by one particular additive normal form in [26] and then further studied in [4, 5]. Finally, relationship between additive normal forms and the fuzzy transform (and Takagi-Sugeno rules consequently) has been discussed in [4].*

Let us recall the definition of the generalized orthogonality [26] which was crucial for the additive normal forms [26, 5].

Definition 1 *We say that $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \dots, n$ fulfill the orthogonality condition if*

$$\bigoplus_{\substack{i=1 \\ i \neq j}}^n \mathbf{A}_i(x) = 1 - \mathbf{A}_j(x). \quad (17)$$

The following lemma characterizes the orthogonality condition as equivalent to the Ruspini one.

Lemma 1 *Fuzzy sets $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \dots, n$ fulfill the orthogonality condition if and only if*

$$\sum_{i=1}^n \mathbf{A}_i(x) = 1. \quad (18)$$

3.1 Subdirect Image Equations

Since the additive model corresponds to fuzzy rules (4) they are expected to be related to the subdirect image inference mechanism. This subsection investigates this relationship via subdirect image fuzzy relation equations.

Theorem 5 *Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Then system (11) is solvable and \mathbf{R}_*^\oplus is a solution.*

Due to Theorem 3 we can state the following corollary of Theorem 5.

Corollary 1 *Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Then $\tilde{\mathbf{R}}_*$ is a solution to system (11).*

Moreover, it can be demonstrated on the basis of the proof of Theorem 5, that \mathbf{R}_*^\oplus is not the only additive model which is a solution to system (11).

Proposition 1 *Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Furthermore, let \blacktriangle be a t-norm such that $* \leq \blacktriangle$. Then the fuzzy relation \mathbf{R}_*^\oplus is a solution to (11).*

Let us briefly summarize results of this subsection. Theorem 5 provides us with an easy-to-check condition certifying a proper performance of an additive model connected to a subdirect image inference mechanism. Moreover, the assumptions refer only to the antecedent fuzzy sets so and their fulfillment can be ensured in advance before an identification process.

Since no solvability is assumed, the theorem impact is even in specifying sufficient solvability condition. This consequently means that $\hat{\mathbf{R}}_*$ is a solution as well. Finally, a wide variety of t-norms can be used in the additive models.

3.2 Direct Image Equations

This subsection focuses on systems of direct image equations. Disjunctive model $\hat{\mathbf{R}}_*$ has been proved to be a solution to system (7) assuming certain conditions. Similarly, we study the additive model on the same system.

Theorem 6 *Let \mathbf{A}_i for $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Furthermore, let $* \leq \otimes$ where \otimes is the Łukasiewicz t-norm. Then system (7) is solvable and \mathbf{R}_*^\oplus is a solution.*

Due to Theorem 1 we can state the following corollary of Theorem 6.

Corollary 2 *Let \mathbf{A}_i for $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Furthermore, let $* \leq \otimes$. Then $\hat{\mathbf{R}}_*$ is a solution to system (7).*

Theorem 6 requires to use a t-norm which is even weaker than the Łukasiewicz one which is already a very weak t-norm, see [18]. So, for practical applications, perhaps only the case when $* = \otimes$ is worth mentioning. In this case, the Łukasiewicz t-norm is used both for, the sup- \otimes composition as an inference method as well as for connecting antecedent and consequent fuzzy sets in the corresponding fuzzy model \mathbf{R}_*^\oplus .

The result is strengthened by the following theorem.

Proposition 2 *Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and \mathbf{A}_i fulfill the Ruspini condition. Furthermore, let \blacktriangle be an arbitrary t-norm and let $* \leq \otimes$. Then $\mathbf{R}_*^{\oplus\blacktriangle}$ is a solution to (7).*

Proposition 2 allows us to deal with a t-norm weaker or equal to the Łukasiewicz one only in the inference mechanism but the fuzzy model can use an arbitrary t-norm \blacktriangle . Similarly to the subdirect image equations, only normality of antecedents and the Ruspini condition were assumed.

4 Multiplicative Models

As we have shown in Section 3, investigation of additive fuzzy models does not bring only results concerning proper usage of such models. Furthermore, it led to new results in the field of systems of fuzzy relation equations. Especially the fact, that fulfilling two, in practice very often required, conditions leads to the solvability of the corresponding fuzzy relation equations. These results are of a high practical importance since they put assumptions only on the antecedents. This enables to identify a fuzzy rule base in such a way to ensure the solvability of an adjoint system of fuzzy relation equations although the consequent fuzzy sets could be arbitrary e.g. identified from data by some algorithm.

It is a well known fact that both systems of fuzzy relation equations are dual [6]. This leads to a natural idea of introducing the following *multiplicative models*

$$\mathbf{R}_*^\otimes(x, y) = \bigotimes_{i=1}^n (\mathbf{A}_i(x) \rightarrow_* \mathbf{B}_i(y)). \quad (19)$$

Multiplicative model \mathbf{R}_*^\otimes is a special case of a model of fuzzy rule base (2) derived from the conjunctive model $\hat{\mathbf{R}}_*$ where the conjunction aggregating particular rules is generally given by a t-norm, in this particular case by the Łukasiewicz t-norm. It is another arithmetic model, dual to the corresponding additive one.

Now we focus on searching for solutions to systems of direct image and subdirect image fuzzy relation equations given by (19).

4.1 Direct Image Equations

Theorem 7 *Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Then system (7) is solvable and \mathbf{R}_*^\otimes is a solution.*

Due to Theorem 1 we can state the following corollary of Theorem 7.

Corollary 3 *Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Then $\hat{\mathbf{R}}_*$ is a solution to (7).*

Analogously to the case of subdirect image equations and additive models, we can use a wide variety of residuation operations in the multiplicative models.

Proposition 3 *Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Furthermore, let \blacktriangle be a t-norm such that $* \leq \blacktriangle$. Then $\mathbf{R}_*^{\otimes\blacktriangle}$ is a solution to (7).*

4.2 Subdirect Image Equations

Theorem 8 Let \mathbf{A}_i for $i = 1, \dots, n$ be normal and fulfill the Ruspini condition. Furthermore, let $* \leq \otimes$. Then \mathbf{R}_*^\otimes is a solution to system (11).

Analogously to the dual case, to the case of an additive fuzzy model and the direct image equation, the Lukasiewicz t-norm is too weak and so $\mathbf{R}_\otimes^\otimes$ is the only case of a practical importance. And analogously to the dual case, Theorem 8 can be strengthened.

Proposition 4 Let \mathbf{A}_i , $i = 1, \dots, n$ be normal and fulfill the orthogonality condition. Furthermore, let \blacktriangle be a t-norm such that $\blacktriangle \leq \otimes$. Then $\mathbf{R}_\blacktriangle^\otimes$ is a solution to (11).

Proposition 4, allows us to deal with a residuation adjoint to a weak t-norm only in the fuzzy relation interpreting a fuzzy rule base but the respective inference machine can be based on an arbitrary residuation operation.

5 Examples

To demonstrate the results introduced in Section 3, let us consider the following example.

Example 1 Let $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \dots, n$ fulfill the Ruspini condition

$$\sum_{i=1}^n \mathbf{A}_i(x) = 1, \quad \forall x \in X. \quad (20)$$

Moreover, let us assume normality of each fuzzy set \mathbf{A}_i .

Let there be a fuzzy rule base (4) where the antecedents are represented by the given fuzzy sets \mathbf{A}_i and the consequents by arbitrary fuzzy sets $\mathbf{B}_i \in \mathcal{F}(Y)$.

Then due to Theorem 5, fuzzy relation

$$\mathbf{R}_\otimes^\oplus(x, y) = \bigoplus_{i=1}^n (\mathbf{A}_i(x) \otimes \mathbf{B}_i(y)) \quad (21)$$

is a solution to the system of fuzzy relation equations

$$\mathbf{A}_i \triangleleft_{\otimes} \mathbf{R} = \mathbf{B}_i, \quad i = 1, \dots, n.$$

Furthermore, this system is also solved by fuzzy relations

$$\hat{\mathbf{R}}_\otimes(x, y) = \bigvee_{i=1}^n (\mathbf{A}_i(x) \otimes \mathbf{B}_i(y)) \quad \text{and}$$

$$\mathbf{R}_\otimes^\oplus(x, y) = \bigoplus_{i=1}^n (\mathbf{A}_i(x) \odot \mathbf{B}_i(y)) = \sum_{i=1}^n \mathbf{A}_i(x) \mathbf{B}_i(y)$$

due to Corollary 1 and Proposition 1, respectively.

Moreover, due to Theorem 6, fuzzy relation (21) is also a solution to the system of fuzzy relation equations

$$\mathbf{A}_i \circ_{\otimes} \mathbf{R} = \mathbf{B}_i \quad i = 1, \dots, n, \quad (22)$$

and due to Proposition 2, a solution to system (22) can be also found in the form of $\mathbf{R}_\otimes^\oplus$.

This means, that fuzzy models $\hat{\mathbf{R}}_\otimes$, $\mathbf{R}_\otimes^\oplus$ and $\mathbf{R}_\otimes^\oplus$ are safe models [27] of fuzzy rules (4) from the fuzzy interpolation point of view when assuming the subdirect image with the Lukasiewicz t-norm as the corresponding inference mechanism. The two latter arithmetic fuzzy models $\mathbf{R}_\otimes^\oplus$ and $\mathbf{R}_\otimes^\oplus$ are safe even in the case of the direct image with the Lukasiewicz t-norm.

To demonstrate the results introduced in Section 4, let us consider the following example.

Example 2 Let $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \dots, n$ fulfill the Ruspini condition. Moreover, let us assume normality of each fuzzy set \mathbf{A}_i .

Let there be a fuzzy rule base (2) where the antecedents are represented by the given fuzzy sets \mathbf{A}_i and the consequents by arbitrary fuzzy sets $\mathbf{B}_i \in \mathcal{F}(Y)$.

Then due to Theorem 7, fuzzy relation

$$\mathbf{R}_\otimes^\otimes(x, y) = \bigotimes_{i=1}^n (\mathbf{A}_i(x) \rightarrow_{\otimes} \mathbf{B}_i(y)) \quad (23)$$

is a solution to the system of fuzzy relation equations

$$\mathbf{A}_i \circ_{\otimes} \mathbf{R} = \mathbf{B}_i, \quad i = 1, \dots, n.$$

Furthermore, this system is also solved by fuzzy relations

$$\hat{\mathbf{R}}_\otimes(x, y) = \bigwedge_{i=1}^n (\mathbf{A}_i(x) \rightarrow_{\otimes} \mathbf{B}_i(y)) \quad \text{and}$$

$$\mathbf{R}_\otimes^\otimes(x, y) = \bigotimes_{i=1}^n (\mathbf{A}_i(x) \rightarrow_{\odot} \mathbf{B}_i(y))$$

due to Corollary 3 and Proposition 3, respectively.

Moreover, due to Theorem 8, fuzzy relation (23) is also a solution to the system of fuzzy relation equations

$$\mathbf{A}_i \triangleleft_{\otimes} \mathbf{R} = \mathbf{B}_i \quad i = 1, \dots, n, \quad (24)$$

and due to Proposition 4, even to the system

$$\mathbf{A}_i \triangleleft_{\odot} \mathbf{R} = \mathbf{B}_i \quad i = 1, \dots, n. \quad (25)$$

This means, that the fuzzy models $\hat{\mathbf{R}}_\otimes$, $\mathbf{R}_\otimes^\otimes$ and $\mathbf{R}_\otimes^\otimes$ are safe models [27] of fuzzy rules (2) from the fuzzy

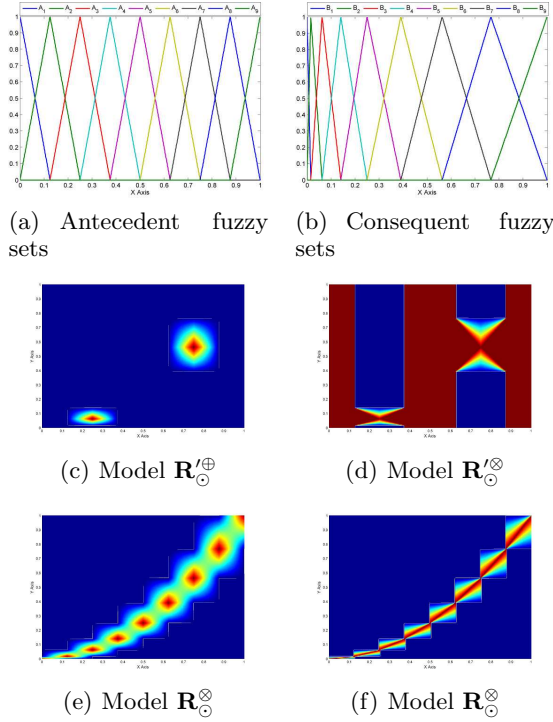


Figure 1: Example of arithmetic fuzzy models.

interpolation point of view when assuming the direct image with the Łukasiewicz t-norm as the corresponding inference mechanism. The last arithmetic fuzzy models $\mathbf{R}_{\otimes}^{\otimes}$ is safe even in the case of the subdirect image with the Łukasiewicz t-norm or the subdirect image with the product t-norm.

In the sequel, we present several figures of the proposed fuzzy models.

Example 3 Let us be given fuzzy pairs $(\mathbf{A}_i, \mathbf{B}_i)$, $i = 1, \dots, 9$ on $\mathcal{F}([0, 1]) \times \mathcal{F}([0, 1])$ approximating the function $y = x^2$. The fuzzy sets \mathbf{A}_i are triangular and form a uniform fuzzy partition of $X = [0, 1]$. The fuzzy sets \mathbf{B}_i are triangular with the kernel points equal to the precise solution of the equation $y = x^2$ for x being the kernel point of the corresponding \mathbf{A}_i and they fulfil the Ruspini condition, see Figures 1(a)–1(b).

Let us observe how the additive and the multiplicative fuzzy models, both with respect to the product t-norm, look if we take into account either only two or all fuzzy pairs.

The fuzzy models $\mathbf{R}_{\oplus}^{\oplus}$ and $\mathbf{R}_{\otimes}^{\otimes}$ take into account only pairs of the fuzzy sets $(\mathbf{A}_3, \mathbf{B}_3)$ and $(\mathbf{A}_7, \mathbf{B}_7)$, the fuzzy models $\mathbf{R}_{\oplus}^{\otimes}$ and $\mathbf{R}_{\otimes}^{\oplus}$ take into account all the given fuzzy pairs.

How the constructed fuzzy models look like (view from above) is displayed on Figure 1.

We can see that the, originally absolutely non-compatible, models converge to each other when the corresponding antecedent fuzzy sets converge to the Ruspini partition.

This observation completes known facts about relationship between partitions, solvability of systems of fuzzy relation equations and possible solutions (cf. Remark 1).

6 Conclusions

Two types of fuzzy rules have been discussed together with corresponding inference mechanisms. New fuzzy models of the rules have been proposed. Additive and multiplicative fuzzy models have been discussed from the fuzzy interpolation point of view, i.e., as possible solutions of corresponding systems of fuzzy relation equations. The crucial role of the Ruspini partition has been proved and visually demonstrated.

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