Pseudo-integral of set-valued functions

Tatjana Grbić
Faculty of Technical Sciences
University of Novi Sad
Trg D. Obradovica 6
Novi Sad, Serbia
tatjana@uns.ns.ac.yu

Ivana Štajner-Papuga
Department of Mathematics
and Informatics
University of Novi Sad
Trg D. Obradovica 4, Novi Sad, Serbia
stajner@im.ns.ac.yu

Ljubo Nedović
Faculty of Technical Sciences
University of Novi Sad
Trg D. Obradovica 6
Novi Sad, Serbia
nljubo@uns.ns.ac.yu

Abstract

An approach to the integration of set-valued functions based on the pseudo-integral has been proposed. Some basic properties of the pseudo-integral of set-valued functions have been shown.

Keywords: Pseudo-operations, Pseudo-integral, Set-valued functions.

1 Introduction

The main topic of this paper is integration of set-valued functions. Over the years, theory of set-valued functions, beside being an important mathematical theory, has become an important tool in several practical areas, specially in economic analysis (problems of individual demand, mean demand, competitive equilibrium, coalition production economies, etc.) (see [5]). Applications of integration of set-valued functions in economy analysis have roots in Aumann's research based on the classical Lebesgue integral ([1]). Some generalizations of this approach that relied on an extension of Lebesgue integral known as $\perp$-integral (see [18]) and on the Choquet integral had been investigated in [19] and [4], respectively.

Results presented in this paper belong to the theory of pseudo-analysis. As a rather new theory, pseudo-analysis has proved itself to be a vast source of powerful tools that are being successfully applied in many mathematical theories as well as in various practical problems (see [2, 6, 7, 9, 11, 12, 13, 14, 15, 16]). Having this in mind, integral proposed here is based on the pseudo-integral ([13]), i.e. pseudo-analysis' counterpart of the classical integral.

Section 2 contains preliminary notions, such as pseudo-operations, semiring, σ-$\oplus$-decomposable measure and pseudo-integral. The construction of the pseudo-integral of set-valued functions is given in the third section. Basic properties of this new type of integral are presented in the Section 4. Some concluding remarks are stated in the Section 5.

2 Preliminary notions

The basic preliminary notions needed in this paper are notions of pseudo-operations and semiring.

Let $[a, b]$ be closed subinterval of $[-\infty, +\infty]$ (in some cases semiclosed subintervals will be considered) and let $\preceq$ be total order on $[a, b]$. A semiring is structure $([a, b], \oplus, \odot)$ such that the following hold:

- $\oplus$ is *pseudo-addition*, i.e., a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element, denoted by $\mathbf{0}$;
- $\odot$ is *pseudo-multiplication*, i.e., a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing $\left(x \preceq y \text{ implies } x \odot z \preceq y \odot z,\ z \in [a, b]_+ = \{x : x \in [a, b],\ \mathbf{0} \preceq x\}\right)$, associative and for which exists a unit element denoted by $\mathbf{1}$;
- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and non idempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by monotone and continuous generator function $g : [a, b] \to [0, +\infty]$, i.e. g-semirings ([8, 10, 12, 13]), form the second class, and semirings with both idempotent operations belong to the third class. More on this structure can be found in [7, 9, 12, 13, 14].

Total order $\preceq$ is closely connected to the choice of the pseudo-addition. If $\oplus$ is an idempotent operation (semirings of the first and the third class), total order is induced in the following way

$$x \preceq y \text{ if and only if } x \oplus y = y,$$

and if $([a,b], \oplus, \odot)$ is a semiring of the second class given by generator g, total order is given by

$$x \preceq y \text{ if and only if } g(x) \leq g(y).$$

In all three cases the strict order $\prec$ has the following form:

$$x \prec y \text{ if and only if } x \preceq y \text{ and } x \neq y.$$

Now, let $([a,b], \oplus, \odot)$ be a semiring and let $([a,b], \oplus)$ and $([a,b], \odot)$ be complete lattice ordered semigrups. Let us suppose that interval $[a,b]$ is endowed with metric d which is compatible with sup and inf and satisfies one of the following conditions:

i) $d\left(x_1 \oplus y_1, x_2 \oplus y_2\right) \leq d\left(x_1, x_2\right) + d\left(y_1, y_2\right),$

ii) $d\left(x_1 \oplus y_1, x_2 \oplus y_2\right) \leq \max\left\{d\left(x_1, x_2\right), d\left(y_1, y_2\right)\right\}.$

Since construction of the pseudo-integral is similar to the construction of Lebesgue integral, necessary notion is also notion of the σ-$\oplus$-measure ([13]).

Let Σ be a σ-algebra of subset of a X. A set function $\mu : \Sigma \to [a,b]_+$ is the *σ-$\oplus$-measure* if

i) $\mu(\emptyset) = \mathbf{0}$,

ii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \bigoplus_{i=1}^{n} \mu(A_i),$

where $(A_i)_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint sets from Σ.

If $\oplus$ is an idempotent operation, then disjointness of sets and condition i) can be omitted.

If $e : X \to [a,b]$ is an elementary function of the following representation

$$e = \bigoplus_{i=1}^{\infty} a_i \odot \chi_{A_i},$$

where $a_i \in [a,b]$, $A_i \in \Sigma$ and χ_A is *the pseudo-characteristic function* of a set A given by

$$\chi_A(x) = \begin{cases} \mathbf{1} & \text{for } x \in A, \\ \mathbf{0} & \text{for } x \notin A, \end{cases} \tag{1}$$

the pseudo-integral of e with respect to σ-$\oplus$-measure μ is

$$\int_X^{\oplus} e \odot d\mu = \bigoplus_{i=1}^{\infty} a_i \odot \mu\left(A_i\right).$$

The pseudo-integral of a bounded measurable function $f : X \to [a,b]$ is

$$\int_X^{\oplus} f \odot d\mu = \lim_{n\to\infty} \int_X^{\oplus} \varphi_n \odot d\mu, \tag{2}$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of elementary functions such that $d\left(\varphi_n(x), f(x)\right) \to 0$ uniformly while $n \to \infty$ and d is previously mentioned metric. Proof of existence of sequence $(\varphi_n)_{n\in\mathbb{N}}$ as well as construction of functions φ_n can be found in [13].

Remark 1 Construction of the pseudo-integral for non idempotent pseudo-addition additionally requires that for each $\varepsilon > 0$ exists a monotone ε-net in $f(X)$ (see [13]).

The pseudo-integral of function f on some arbitrary subset A of X is given by

$$\int_A^{\oplus} f \odot d\mu = \int_X^{\oplus} (f \odot \chi_A) \odot d\mu, \tag{3}$$

where χ_A is the pseudo-characteristic function (1).

3 Pseudo-integral of set-valued functions

The definition of set-valued pseudo-integral will be given in this section.

Let $([a,b], \oplus, \odot)$ be a semiring as described in the previous section. Let $f : X \to [a,b]_+$ be a measurable real-valued function, i.e., sets of the form $\{x | \alpha \prec f(x)\}$ are measurable for all $\alpha \in [a,b]_+$. Further more, let us suppose that function f is integrable with respect to pseudo-integral given by (2), that is, let $\int_X^{\oplus} f \odot d\mu$ exist as a finite value in the sense of semiring $([a,b], \oplus, \odot)$. The family of all measurable and integrable functions with respect to measure μ and pseudo-integral (2) will be denoted with $L^1_{\oplus}(\mu)$.

Let $\mathcal{F}$ be the class of all closed subsets of $[a,b]_+$. *A set-valued function* is a function from X to $\mathcal{F} \backslash \{\emptyset\}$. Further on, by a measurable set-valued function we shall consider set-valued functions with measurable graph.

Definition 2 *Let F be a set-valued function.* The pseudo-integral *of F on $A \in \Sigma$ is*

$$\int_A^{\oplus} F \odot d\mu = \left\{ \int_A^{\oplus} f \odot d\mu \,\middle|\, f \in S(F) \right\}, \tag{4}$$

where $S(F)$ is the family of μ-a.e. measurable selections of F, i.e.

$$S(F) = \left\{ f \in L^1_{\oplus}(\mu) \,|\, f(x) \in F(x) \;\; \mu - a.e. \text{ on } X \right\}.$$

Specially, when $\int^{\oplus}$ coincides with Lebesgue integral (see ([12, 13]), integral (4) is the classical Aumann's integral ([1]).

The first question that arises is the question of pseudo-integrability of set-valued functions.

Definition 3 *A set-valued function $F : X \to \mathcal{F}\backslash\{\emptyset\}$ is pseudo-integrable on $A \in \Sigma$ if*

$$\int_A^{\oplus} F \odot d\mu \neq \emptyset.$$

In order to address this issue, the following property of set-valued functions has to be defined:

Definition 4 *A set-valued function F is* pseudo-integrably bounded *if there is a function $h \in L^1_{\oplus}(\mu)$ such that:*

i) $\bigoplus_{\alpha \in F(x)} \alpha \preceq h(x)$, *for idempotent pseudo-addition,*

ii) $\sup_{\alpha \in F(x)} \alpha \preceq h(x)$, *for pseudo-addition given by increasing generator g,*

iii) $\inf_{\alpha \in F(x)} \alpha \preceq h(x)$, *for pseudo-addition given by decreasing generator g.*

Sufficient condition for pseudo-integrability of set-valued function F is given by the following proposition.

Proposition 5 *If F is a pseudo-integrably bounded set-valued function, then F is pseudo-integrable.*

Proof. Let F be a pseudo-integrably bounded set-valued function and let h be a function from Definition 4. Let f be a selection of F, i.e., $f(x) \in F(x)$ μ-a.e. on X. It can be easily shown that, under give assumptions, the following holds almost everywhere

$$f \preceq h.$$

Now, properties of the pseudo-integral will insure us

$$\int_X^{\oplus} f \odot d\mu \preceq \int_X^{\oplus} h \odot d\mu$$

(see [13]). Since we assumed that h is integrable function, due to the properties of semiring $([a,b], \oplus, \odot)$ function f is also integrable, i.e., set (4) is not empty. □

Example 6
(a) Let us consider semiring of the first class, e.g. $([a,b], \max, +)$. Then, σ-$\oplus$-measure μ is given by some function $l : \mathbb{R} \to [a,b]$ as $\mu(A) = \sup_{x \in A} l(x)$ (see [11, 12, 13]). In this case, pseudo-integral of some set-valued function F has the following form

$$\int_A^{\oplus} F \odot d\mu = \left\{ \sup_{x \in A} \left(f(x) + l(x)\right) \mid f \in S(F) \right\}.$$

(b) If $([a,b], \oplus, \odot)$ is a semiring of the second class pseudo-operations are given by generating function $g : [a,b] \to [0, \infty]$ as

$$x \oplus y = g^{-1}\left(g(x) + g(y)\right) \text{ and } x \odot y = g^{-1}\left(g(x)g(y)\right)$$

(see [6, 8, 10, 12, 13, 15]), therefore pseudo-integral of some set-valued function F is

$$\int_{[c,d]}^{\oplus} F \odot d\mu = \left\{ g^{-1}\left(\int_{[c,d]} g \circ f \, d(g \circ \mu) \right) \middle| f \in S(F) \right\}.$$

(c) Semiring $([a,b], \min, \max)$ is a semiring of the third class. Now, σ-$\oplus$-measure μ is given by some function $l : \mathbb{R} \to [a,b]$ as $\mu(A) = \inf_{x \in A} l(x)$. In this case, pseudo-integral of some set-valued function F has the following form

$$\int_A^{\oplus} F \odot d\mu = \left\{ \inf_{x \in A} \left(\max(f(x), l(x))\right) \mid f \in S(F) \right\}.$$

Remark 7 Problem of pseudo-integral of set-valued function for semiring of the second class given by increasing continuous generator had been investigated in [3].

4 Basic properties of set-valued pseudo-integral

Some basic properties of integral given by Definition 4 will be presented in this section.

Proposition 8 *Let F be pseudo-integrable set-valued function, F_1 and F_2 pseudo-integrably bounded set-valued functions and let $A, B \in \Sigma$.*

i) If $A \subset B$ then $\int_A^{\oplus} F \odot d\mu \preceq \int_B^{\oplus} F \odot d\mu$.

ii) If $F_1 \preceq F_2$ then $\int_X^{\oplus} F_1 \odot d\mu \preceq \int_X^{\oplus} F_2 \odot d\mu$.

iii) If $\mu(A) = \mathbf{0}$ then $\int_A^{\oplus} F \odot d\mu = \{\mathbf{0}\}$.

iv) If $\mathbf{0} \prec \alpha$ then

$$\int_X^{\oplus} (\alpha \odot F) \odot d\mu = \alpha \odot \int_X^{\oplus} F \odot d\mu.$$

Remark 9 Relation "less or equal" applied on sets from $\mathcal{F}$ in Proposition 8 is given in the following sense: if $C, D \in \mathcal{F}$ and $C \preceq D$ then

for all $x \in C$ there exists $y \in D$ such that $x \preceq y$

and

for all $y \in D$ there exists $x \in C$ such that $x \preceq y$

Proof of Proposition 8.

i) Let us suppose that $x \in \int_A^{\oplus} F \odot d\mu$. Then, by Definition 4, there exists $f \in S(F)$ such that $x = \int_A^{\oplus} f \odot d\mu$. Now, (3) gives us following

$$x = \int_X^{\oplus} (f \odot \chi_A) \odot d\mu.$$

Since it can be easily shown that in our case $f \odot \chi_A \preceq f \odot \chi_B$, properties of pseudo-integral will ensure

$$\int_X^{\oplus} (f \odot \chi_A) \odot d\mu \preceq \int_X^{\oplus} (f \odot \chi_B) \odot d\mu.$$

Obviously,

$$\int_X^{\oplus} (f \odot \chi_B) \odot d\mu = \int_B^{\oplus} f \odot d\mu \in \int_B^{\oplus} F \odot d\mu,$$

therefor there exists

$$y = \int_B^{\oplus} f \odot d\mu \in \int_B^{\oplus} F \odot d\mu,$$

such that $x \preceq y$.

Proof that for $y \in \int_B^{\oplus} F \odot d\mu$ exists $x \in \int_A^{\oplus} F \odot d\mu$ such that $x \preceq y$ is analogous to the previously shown part "for all $x \in \int_A^{\oplus} F \odot d\mu$ exists $y \in \int_B^{\oplus} F \odot d\mu$ such that $x \preceq y$".

ii) As in i), it has to be proved that for all $x \in \int_X^{\oplus} F_1 \odot d\mu$ exists $y \in \int_X^{\oplus} F_2 \odot d\mu$ such that $x \preceq y$ and viceversa, i.e., that for all $y \in \int_X^{\oplus} F_2 \odot d\mu$ exists $x \in \int_X^{\oplus} F_1 \odot d\mu$ such that $x \preceq y$.

Since F_1 is pseudo-integrably bounded set-valued function, it is pseudo-integrable set-valued functions, i.e., $\int_X^{\oplus} F_1 \odot d\mu \neq \emptyset$, therefor let $x = \int_X^{\oplus} f \odot d\mu$ for some $f \in S(F_1)$. Let us suppose that $\oplus$ is max or strict pseudo-operation given by increasing generator. Now, having in mind that $F_1(u) \preceq F_2(u)$ for all $u \in X$, following function can be defined:

$$k(u) = \sup \{ v \mid v \in F_2(u),\ f(u) \preceq v \}.$$

Obviously $f \preceq k$ and, sice $\mathcal{F}$ is a family of closed sets, $k(u) \in F_2(u)$. Based on the assumption that F_2 is also pseudo-integrably bounded set-valued function and on properties of pseudo-integral, it can be easily shown that $k \in L^1_{\oplus}$ and that required y is of the form

$$y = \int_X^{\oplus} k \odot d\mu \in \int_X^{\oplus} F_2 \odot d\mu.$$

If pseudo-addition in question is min or strict operation given by decreasing generator, for given $x = \int_X^{\oplus} f \odot d\mu$ required $y \in \int_X^{\oplus} F_2 \odot d\mu$ can be obtained through function

$$k(u) = \inf \{ v \mid v \in F_2(u),\ f(u) \preceq v \}$$

as $y = \int_X^{\oplus} k \odot d\mu$.

Proof of part "for all $y \in \int_X^{\oplus} F_2 \odot d\mu$ there exists $x \in \int_X^{\oplus} F_1 \odot d\mu$ such that $x \preceq y$" if $\oplus$ is max or strict pseudo-operation given by increasing generator is based on function

$$k_2(u) = \inf \{ v \mid v \in F_1(u),\ v \preceq f_2(u) \},$$

i.e., for $y = \int_X^{\oplus} f_2 \odot d\mu,\ f_2 \in S(F_2)$, required x is $\int_X^{\oplus} k_2 \odot d\mu$. Analogously, if $\oplus$ is min or strict pseudo-operation given by decreasing generator, then proof is based on function

$$k_2(u) = \sup \{ v \mid v \in F_1(u),\ v \preceq f_2(u) \},$$

i.e., for $y = \int_X^{\oplus} f_2 \odot d\mu,\ f_2 \in S(F_2)$, required x is $\int_X^{\oplus} k_2 \odot d\mu$.

Statements iii) and iv) follow directly from Definition 4 and properties of the pseudo-integral (see [11, 13]). □

Also, for set-valued pseudo-integrals pseudo-convex property can be easily shown, where by pseudo-convex set following is considered:

$$\begin{array}{c}\text{set } A \subseteq [a,b] \text{ is } \textit{pseudo-convex} \text{ if for} \\ x, y \in A \text{ and } \alpha, \beta \in [a,b]_+ \text{ where } \alpha \oplus \beta = \mathbf{1} \text{ holds} \\ \alpha \odot x \oplus \beta \odot y \in A. \end{array} \tag{5}$$

Proposition 10 *Let F be a measurable set-valued function such that $F(x)$ is pseudo-convex for μ-a.e. $x \in X$. Then, $\int_X^{\oplus} F \odot d\mu$ is a pseudo-convex subset of $[a,b]_+$.*

Proof. Proof follows directly from property of pseudo-convex sets (5) and Definition 4. □

5 Conclusion

Topic presented here belongs to a contemporary field of mathematics that has been successfully applied in economic analysis. Further research of this combination of set-valued functions theory and pseudo-analysis will be directed to the problems of convergence for sequences of set-valued pseudo-integrals and possible applications.

Acknowledgement. This paper has been supported by the project MNZŽSS 144012. The second and the third authors are supported by the project "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications" of the Academy of Sciences and Arts of Vojvodina supported by Provincial Secretariat for Science and Technological Development of Vojvodina.

References

[1] R. J. Aumann, Integrals of set-valued functions, Journal of Mathematical Analysis and Applictions 12 (1965) 1–12.

[2] P. Benvenuti, R. Mesiar and D. Vivona, Monotone Set Functions-Based Integrals, in: E. Pap (Ed.), *Handbook of Measure Theory, Volume II.* Elsevier, Amsterdam, 2002, pp. 1329–1379.

[3] C. Guo, D. Zhang, On set-valued fuzzy measures, Information Sciences 160 (2004) 13–25.

[4] L. C. Jang, B. M. Kil, Y. K. Kim, J. S. Kwon, Some properties of Choquet integrals of set-valued functions, Fuzzy Sets and Systems 91 (1997) 95–98.

[5] E. Klein, A C. Thompson, Theory of Correspondences, A Wi ley-Interscience Publication, New York, 1984.

[6] E.P. Klement, R. Mesiar and E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.

[7] V. N. Kolokoltsov and V.P. Maslov, Idempotent Analysis and Its Applications, Kluwer Academic Publishers, Dordrecht, 1997.

[8] A. Marková-Stupňanová, A note on g-derivative and g-integral, Tatra Mt. Math. Publ. 8 (1996) 71–76.

[9] V.P. Maslov, S.N. Samborskij, Idempotent Analysis, American Mathematical Society, Providence, 1992.

[10] R. Mesiar, Pseudo-linear integrals and derivatives based on a generator g, Tatra Mt. Math. Publ. 8 (1996) 67–70.

[11] E. Pap, An integral generated by decomposable measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20 (1990) 135–144.

[12] E. Pap, g-calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 23 (1993) 145–156.

[13] E.Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[14] E. Pap,Pseudo-additive measures and their applications, in: E. Pap (Ed.), *Handbook of Measure Theory, Volume II.* Elsevier, Amsterdam, 2002, pp. 1403-1465.

[15] E. Pap, Applications of the generated pseudo-analysis to nonlinear partial differential equations, Contemporary Mathematics 377 (2005) 239–259.

[16] E. Pap, D. Vivona, Non-commutative and non-associative pseudo-analysis and its applications on nonlinear partial differential equations, J. Math. Anal. Appl. 246/2 (2000) 390–408.

[17] D. Zhang, C. Guo, Integrals of set-valued functions for $\perp$-decomposable measures, Fuzzy Sets and Systems 78 (1996) 341–346.

[18] S. Weber, Conditinal measures and their applications to fuzzy sets, Fuzzy Sets and Systems 42 (1991) 73–85.

[19] D. Zhang, C. Guo, Integrals of set-valued functions for $\perp$-decomposable measures, Fuzzy Sets and Systems 78 (1996) 341–346.