

Algorithms for the Construction of Digital Convex Fuzzy Hulls

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Abstract

We consider a digital fuzzy set placed in Oxy -plane, with support which is a set of digital points (centroids) in \mathbb{Z}^2 . We consider two kinds of convexity of a fuzzy set, namely quasi convexity and strong convexity, and we propose two algorithms for the construction of both kinds of convex hull of a digital fuzzy set.

Keywords: fuzzy set, digital fuzzy set, quasiconvexity, convexity, DL-convex hull, digital quasiconvex fuzzy hull, digital convex fuzzy hull.

1 Introduction

Convexity is one of the basic properties of sets and shapes. Convex hull $H(S)$ of a set S is defined as the smallest convex set containing S . Convex hull is used as a tool in many different areas. In robotics, for example, it is used in collision detection, while in shape analysis, the difference between the shape and its convex hull is used as a shape descriptor.

The problem of the computation of the convex hull has been considered for a long time in many research areas, such as computational geometry, image analysis, and fuzzy set theory. In each of these areas, the problem has specific properties. In computational geometry, a finite set S of points in \mathbb{R}^n is given, so points have arbitrary real coordinates [6]. In image analysis, points are restricted to have integer coordinates. The notion of the convex hull $H(I)$ of an image I depends on the way a convex set in \mathbb{Z}^n is defined. Most often, T-convexity, L-convexity, and DL-convexity are used [2]. In fuzzy set theory, the notion of convexity is also not unique. Most often, quasiconvexity, (strong) convexity, and α -convexity are considered. Moreover, the universal set of a fuzzy set A can be either \mathbb{R}^n or \mathbb{Z}^n . Here, we consider the problem of the computation

of the quasiconvex hull $\mathcal{QH}(A)$ and the (strong) convex hull $\mathcal{H}(A)$ of a fuzzy set A , with support $\text{supp}(A)$ which is a digital set, and we propose algorithms for computation of $\mathcal{QH}(A)$ and $\mathcal{H}(A)$ in 2D. Since a gray-scale image I may be considered as a fuzzy set with a discrete domain, our algorithms can be used to find convex hulls of 2D gray-scale images.

2 Background Notions

A set S in Euclidean space is convex if for any two points P and Q we have:

$$\text{for any } \lambda \in [0, 1], \lambda P + (1 - \lambda)Q \in S.$$

The convex hull $H(S)$ of a set S is the intersection of all convex sets that contain S :

$$H(S) = \bigcap_{i \in I} \{S_i | S \subset S_i, S_i \text{ is convex}\}.$$

A set S is convex iff $H(S) = S$.

Euclidean plane Oxy (\mathbb{R}^2) can be digitized using specific regular tessellation; each square is called a pixel. Set \mathcal{D} of all pixels (of \mathbb{R}^2) is called a digital space:

$$\mathcal{D} = \{(i - 0.5, i + 0.5) \times (j - 0.5, j + 0.5) | (i, j) \in \mathbb{Z}^2\}.$$

Set R (or region) in the digital space is the union of a set of pixels, and we call it a digital set. The centroid of a pixel is the point of intersection of the two diagonals of the pixel. The union of the centroids of the pixels of R , denoted by R' , may be considered as the lattice point or simply a point representation of R . Conversely, R may be called the region representation of R' .

Clearly, there exists a 1-1 correspondence δ between digital space \mathcal{D} and set \mathbb{Z}^2 , and thus between R and R' . We call δ digitalization.

Definition 1 *Digital straight line segment P_1P_2 is the set of all pixels that are intersected by Euclidean*

straight line segment p_1p_2 , where p_1, p_2 are centroids (digital points) of pixels P_1, P_2 .

Definition 2 A digital region R represented by R' is called *DL-convex* if for any two digital points p_1, p_2 belonging to R' , there is a digital straight line segment between P_1 and P_2 whose pixels all belong to R .

Definition 3 $H_{DL}(R)$, the *DL-hull* of R is the minimal *DL-convex* set that contains R .

An algorithm for the computation of a *DL-hull* is given in [2].

Let us present some basic definitions and theorems related to fuzzy sets and their convexity, and extend them on digital fuzzy sets. For more details, see [3, 4, 7].

Definition 4 A fuzzy set A defined on the universal set X is a set of ordered pairs: $A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A : X \rightarrow [0, 1]$ is the membership function of A in X .

Definition 5 The support of a fuzzy set A is the set

$$\text{supp}(A) = \{x \in X | \mu_A(x) > 0\}.$$

Definition 6 An α -cut of a fuzzy set A , for $\alpha \in [0, 1]$, is the set:

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}.$$

Definition 7 A fuzzy set A is a subset of a fuzzy set B , denoted $A \subseteq B$, iff

$$\mu_A(x) \leq \mu_B(x),$$

for all $x \in X$.

Definition 8 A fuzzy set A of a set \mathbb{R}^n , given by the membership function μ_A , is called *quasiconvex* if

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

holds for every $x_1, x_2 \in \text{supp}(A)$, and $\lambda \in [0, 1]$ (see [1], [8]).

Theorem 1 A fuzzy set A is quasiconvex if and only if its α -cuts are empty or convex sets for each $\alpha \in [0, 1]$ (see [5]).

Definition 9 A fuzzy subset A of a set \mathbb{R}^n , given by the membership function μ_A , is called *convex* if

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda\mu_A(x_1) + (1 - \lambda)\mu_A(x_2)$$

holds for every $x_1, x_2 \in \text{supp}(A)$, and $\lambda \in [0, 1]$.

Theorem 2 A fuzzy set A is convex if and only if the fuzzy hypograph of A , denoted by $f.\text{hyp}(A)$, is a convex set, where

$$f.\text{hyp}(A) = \{(x, t) | x \in \mathbb{R}^n, t \in (0, \mu_A(x))\}$$

(see [1]).

If A is a convex fuzzy set, then for every $\alpha \in [0, 1]$, α -cut of A is convex (or empty), but the converse does not hold in general.

For example, a fuzzy subset A of \mathbb{R} , given by its membership function $\mu_A(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$ is quasiconvex, but it is not convex.

Definition 10 A (quasi) convex fuzzy hull, ($qconv(A)$) $conv(A)$ of a fuzzy set A is the smallest (with respect to relation \subseteq) (quasi) convex fuzzy set containing A .

If A is a convex fuzzy set, then the convex fuzzy hull of A is equal to A .

Definition 11 A digital fuzzy set is a fuzzy set which is defined on digital space \mathcal{D} , i.e., on its point representation $\mathcal{D}' = \mathbb{Z}^2$.

Definition 12 Let $A \subseteq \mathcal{D}$ (or $A' \subseteq \mathbb{Z}^2$), be a digital fuzzy set. A digital quasiconvex fuzzy hull $\mathcal{QH}(A)$ of A is the smallest digital fuzzy set, given by the membership function μ , such that

$$\text{supp}(\mathcal{QH}(A)) = H_{DL}(\text{supp}(A)), \quad (1)$$

$$\mu(\lambda p_1 + (1 - \lambda)p_2) \geq \min\{\mu(p_1), \mu(p_2)\} \quad (2)$$

holds for every $p_1, p_2 \in H'_{DL}(\text{supp}(A))$, and for all $\lambda \in [0, 1]$ such that $\lambda p_1 + (1 - \lambda)p_2 \in \mathbb{Z}^2$.

Definition 13 Let $A \subseteq \mathcal{D}$ (or $A' \subseteq \mathbb{Z}^2$), be a digital fuzzy set. A digital convex fuzzy hull $\mathcal{H}(A)$ of A is the smallest digital fuzzy set, given by the membership function μ , such that

$$\text{supp}(\mathcal{H}(A)) = H_{DL}(\text{supp}(A)), \quad (3)$$

$$\mu(\lambda p_1 + (1 - \lambda)p_2) \geq \lambda\mu(p_1) + (1 - \lambda)\mu(p_2) \quad (4)$$

holds for every $p_1, p_2 \in H'_{DL}(\text{supp}(A))$, and for all $\lambda \in [0, 1]$ such that $\lambda p_1 + (1 - \lambda)p_2 \in \mathbb{Z}^2$.

3 Algorithms

Let A be a digital fuzzy set, with membership function μ_A . Let us construct the *DL-hull* $H_{DL}(\text{supp}(A))$, using the algorithm given in [2]. Let

$$H'_{DL}(\text{supp}(A)) = \mathcal{A} = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\},$$

$n \geq 2$, be the digitalization of $H_{DL}(\text{supp}(A))$.

For $A_k(x_k, y_k) \in \mathcal{A}$, let $\tilde{A}_k(x_k, y_k, \mu(A_k))$, where $\mu(A_k) = \mu_A(A_k) = \mu_k$. The sets \mathcal{A} and $\tilde{\mathcal{A}} = \{\tilde{A}_1(x_1, y_1, \mu(A_1)), \dots, \tilde{A}_n(x_n, y_n, \mu(A_n))\}$, are the input sets to both our algorithms.

3.1 Algorithm for the Construction of a Digital Quasiconvex Fuzzy Hull $\mathcal{QH}(A)$

Let us present the algorithm for computing a digital quasiconvex fuzzy hull of a digital fuzzy set A . As explained above, the input to the algorithm is the set \mathcal{A} .

The idea of the algorithm is to process the input points iteratively, in the non-increasing order of their membership values, and to construct the set \mathcal{AFH} , by forcing α -cuts of \mathcal{AFH} to be convex sets. Also, we shall prove that by using this algorithm, the digital quasiconvex fuzzy hull $\mathcal{QH}(A)$ of A can be obtained.

Algorithm 1 (for computing digital quasiconvex fuzzy hull):

step 0: Initialize

$\mathcal{E} = \emptyset$ (\mathcal{E} is the set of extremal points),

$\mathcal{A}_H = \emptyset$ (\mathcal{A}_H is the set of internal - nonextremal points).

step 1: Order set \mathcal{A} by non increasing order of membership function μ and rename the points of \mathcal{A} as A_1, A_2, \dots, A_n , so that $\mu(A_1) \geq \mu(A_2) \geq \dots \geq \mu(A_n)$.

step 2: Let $p = p(A_1, A_2)$ be the line determined with points A_1 and A_2 . If $\mu(A_1) = \mu(A_2)$, go to step 3'. Otherwise, (if $\mu(A_1) > \mu(A_2)$), go to step 3''.

step 3': Find all points of set $p \cap \mathcal{A}$ whose membership function is equal to $\mu(A_1) = \mu(A_2)$. Find end points E'_1 and E''_1 of that set, remove them from \mathcal{A} , and store them in \mathcal{E} . Remove from \mathcal{A} all points $A_p \in \mathcal{A}$, which belong to line segment $E'_1 E''_1$, such that $\mu(A_p) < \mu(E'_1)$, and store them in \mathcal{A}_H , with changed membership function $\mu(A_p) = \mu(E'_1) = \mu(E''_1)$ (see Figure 1). Go to step 5.

step 3'': Remove the point A_1 from \mathcal{A} and store it in \mathcal{E} , with unchanged membership function. Set $E''_1 \equiv E'_1 \equiv A_1$ and $\mu(E''_1) = \mu(E'_1) = \mu(A_1)$. Rename points of set \mathcal{A} so that A_1 is the point with the largest membership function, next point is A_2 , etc.

step 4: Find all points of set $p \cap \mathcal{A}$ which have equal membership function as point A_1 , together with points E'_1 and E''_1 (E'_{i-1} and E''_{i-1}) (together with only one

of those two points, if they are equal).

Denote with E'_2 and E''_2 (E'_i and E''_i) such points from all those points (if they exist), that any other point A_k from $p \cap \mathcal{A}$ either coincides with one of them, or is between them. We also take that the points of p are in the order $E'_2 \preceq E'_1 \preceq E''_1 \preceq E''_2$ ($E'_i \preceq E'_{i-1} \preceq E''_{i-1} \preceq E''_i$), where the relation \preceq is one of relations: \equiv (congruence of points) or $-$ (be between two points).

Store the points E'_2 and E''_2 (E'_i and E''_i) in set \mathcal{E} , if they are not already in \mathcal{E} . If one of them is already an extreme point, assign to it the membership function of the other point, i.e. $\mu(E'_i) = \mu(E''_i)$. For example, if $E''_3 \equiv E'_2 \in \mathcal{E}$, we take that $\mu(E''_3) = \mu(E'_3)$ (see Figure 1).

Remove other points $A_k \in p \cap \mathcal{A}$ from \mathcal{A} , and store them in \mathcal{A}_H .

Remove from \mathcal{A} all points $A_q \in \mathcal{A}$, which belong to the line segment $E'_2 E''_2$ ($E'_i E''_i$), such that $\mu(A_q) < \mu(E'_2)$ ($\mu(A_q) < \mu(E'_i)$), and store them in \mathcal{A}_H , with changed membership function $\mu(A_q) = \mu(E'_2) = \mu(E''_2)$ ($\mu(A_q) = \mu(E'_i) = \mu(E''_i)$).

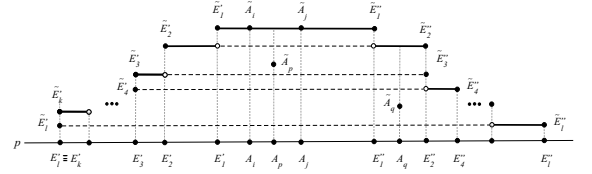


Figure 1: First steps of Algorithm 1. Points of the current α -cut are all on a line.

step 5: Rename points of set \mathcal{A} so that A_1 is the point with the largest membership function, next point is A_2 , etc. If $A_1 \in p$, go to step 4, where input is again A_1 , and output is E'_i and E''_i . This iterative cycle is repeated as long as the previous condition is satisfied. Otherwise, denote with K_1 and K_2 the two end points from $p \cap \mathcal{E}$ (E'_i i E''_i in Figure 1). Introduce the set $\mathcal{K}_0 = \{K_1, K_2\}$.

step 6: Join the point A_1 with points K_1 and K_2 from set $\mathcal{K}_0 \subset \mathcal{E}$. Remove from \mathcal{A} all points A_k that are in the obtained triangle (together with point A_1), and store them in \mathcal{A}_H , with changed membership function $\mu(A_k) = \mu(A_1)$. Store in \mathcal{E} the point A_1 with its membership function. This situation is illustrated in Figure 2.

Set $K_3 \equiv A_1$ and $\mu(K_3) = \mu(A_1)$ and store point K_3 in \mathcal{K}_0 . In that way we get the set $\mathcal{K}_1 = \mathcal{K}_0 \cup \{K_3\}$. Mark the derived triangle with $\mathcal{M}(\mathcal{K}_1)$.

step 7: In the next iteration (and those after if they are needed) we take the first point (from the remain-

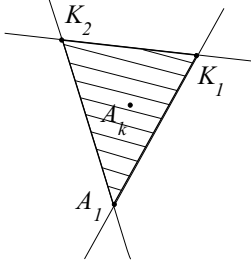


Figure 2: A_1 does not belong to the line p .

ing points, if they exist) from set \mathcal{A} . Because of the renaming of points of the set \mathcal{A} , that point is A_1 .

If $\mathcal{A} = \emptyset$, then go to step 8.

Store in \mathcal{E} the point A_1 with unchanged membership function, and remove A_1 from \mathcal{A} . If on the derived line segments or in the interior of the derived polygon in current iteration there are left some of the extreme points with the same membership function $\mu(A_1)$, remove them from set \mathcal{E} and store them in set \mathcal{A}_H with unchanged membership function.

From points of set $\mathcal{K}_2 = \{K_1, K_2, K_3\}$ ($\mathcal{K}_{r-1} = \{K_1, K_2, K_3, \dots, K_{i-1}, K_i, K_{i+1}, \dots, K_{j-1}, K_j, K_{j+1}, \dots, K_m\}$) and point A_1 we make another set \mathcal{K}_4 (\mathcal{K}_r), which contains vertices of the convex hull of set $\mathcal{K}_3 \cup \{A_1\}$ ($\mathcal{K}_{r-1} \cup \{A_1\}$). In Figure 3, $\mathcal{K}_4 = \{K_1, K_2, K_3, A_1\}$, in Figures 4 and 5, $\mathcal{K}_4 = \{K_1, K_2, A_1\}$. In derived set \mathcal{K}_4 (\mathcal{K}_r) we also rename indexes and denote point A_1 with K_4 in the case like in Figure 3, or with K_3 in the cases as in Figures 4 and 5 (with K_i for some index i in the general case).

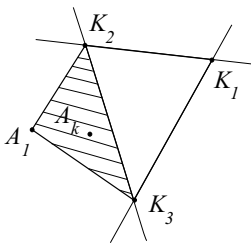


Figure 3: Triangle $\mathcal{M}(\mathcal{K}_1)$ is extended with point A_1 into quadrangle $K_1K_2A_1K_3$.

Denote with $\mathcal{M}(\mathcal{K}_4)$ ($\mathcal{M}(\mathcal{K}_r)$) polygon determined with points from \mathcal{K}_4 (\mathcal{K}_r). Remove from \mathcal{A} all points A_k from $\mathcal{A} \cap (\mathcal{M}(\mathcal{K}_4) \setminus \mathcal{M}(\mathcal{K}_3))$ ($\mathcal{A} \cap (\mathcal{M}(\mathcal{K}_r) \setminus \mathcal{M}(\mathcal{K}_{r-1}))$), and store them in \mathcal{A}_H , with improved membership function, from $\mu(A_k)$ to $\mu(A_1)$, if that value is less than $\mu(A_1)$ (it is possible that $\mu(A_k) < \mu(A_1)$ or $\mu(A_k) = \mu(A_1)$).

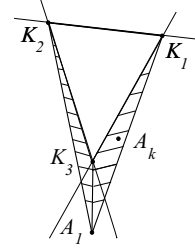


Figure 4: Triangle $\mathcal{M}(\mathcal{K}_1)$ is extended with point A_1 into triangle $K_1K_2A_1$.

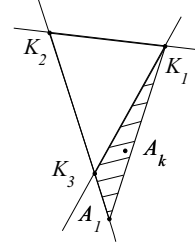


Figure 5: Triangle $\mathcal{M}(\mathcal{K}_1)$ is extended with point A_1 into triangle $K_1K_2A_1$.

If point A_1 is colinear with some two neighboring points from \mathcal{K}_3 (\mathcal{K}_{r-1}), like in Figure 5 and the middle point of that three points (in Figure 5, it is K_3) has membership function equal to $\mu(A_1)$, remove it from \mathcal{E} and store it in \mathcal{A}_H .

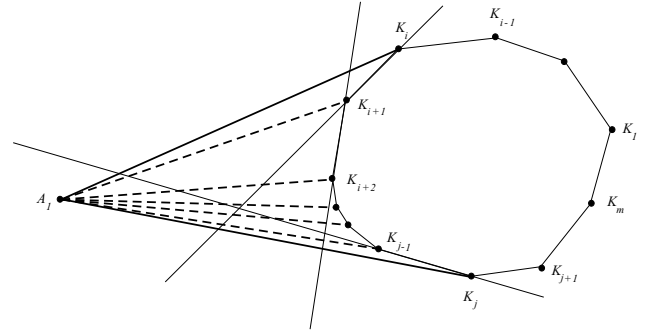


Figure 6: A_1 does not belong to the convex polygon $\mathcal{M}(\mathcal{K}_r)$. A new convex polygon is constructed.

step 8: $\mathcal{A}_{FH} = \mathcal{E} \cup \mathcal{A}_H$. The end.

Let us denote by $\delta^{-1}(\mathcal{A}_{FH})$ the digital fuzzy set which corresponds to a fuzzy set \mathcal{A}_{FH} , i.e., the digital fuzzy set whose support consists of a set of pixels $P = \delta^{-1}(p)$, where $p \in \text{supp}(\mathcal{A}_{FH})$, with the same membership function ($\mu(P) = \mu(p)$).

Theorem 3 *The digital fuzzy set $\delta^{-1}(\mathcal{A}_{FH})$, derived by the previous algorithm, is the digital quasiconvex*

fuzzy hull $\mathcal{QH}(A)$ of set A .

Proof. Any α -cut of the set $\mathcal{A}_{\mathcal{FH}}$ is by construction either a point or a line segment (this is the case when this line segment belongs to the line $p = p(A_1, A_2)$ from the algorithm), or a polygon $\mathcal{M}(K_r)$ with its interior. In both cases, this α -cut is a convex set, and so by Theorem 1 the set $\mathcal{A}_{\mathcal{FH}}$ is a convex set. By construction, $\mathcal{A}_{\mathcal{FH}}$ is the smallest fuzzy set such that (2) holds. $\text{supp}(\mathcal{A}_{\mathcal{FH}})$ is equal to the support of \mathcal{A} , which is by construction (in the preprocessing step) the digitalization of the DL -hull of the support $\text{supp}(A)$ of the fuzzy set A . \square

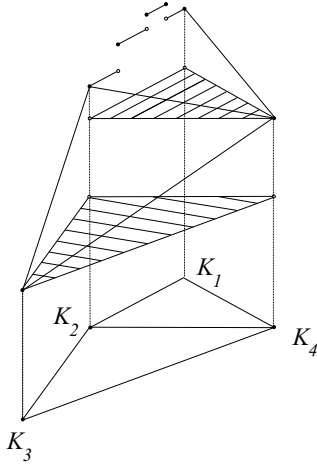


Figure 7: Illustration of the Algorithm 1.

3.2 Algorithm for the Construction of a Digital Fuzzy Convex Hull $\mathcal{H}(A)$

Let us present the algorithm for computing a digital convex fuzzy hull $\mathcal{H}(A)$ of a digital fuzzy set A . As already explained, the input to the algorithm is the set \mathcal{A} .

Let us denote by \mathcal{A}^0 the set \mathcal{A} embedded in \mathbb{R}^3 , i.e., $\mathcal{A}^0 = \{A_1(x_1, y_1, 0), \dots, A_n(x_n, y_n, 0)\}$, and let $\tilde{\mathcal{A}} = \{\tilde{A}_1(x_1, y_1, \mu_1), \dots, \tilde{A}_n(x_n, y_n, \mu_n)\}$, with $\mu_k = \mu(A_k)$, $k = 1, \dots, n$.

The idea of the algorithm is to adapt the gift-wrapping method for the computation of the (discrete) convex hull $H(S)$ of a set S of points in \mathbb{R}^3 . When we apply the adapted algorithm on the set $S = \tilde{\mathcal{A}} \cup \mathcal{A}^0$, we obtain its convex hull $H(S)$, and we simultaneously construct the fuzzy set $\mathcal{A}_{\mathcal{FH}}$. The adaptation is based on the fact that we do not need the bottom and the side facets of the convex hull $H(S)$ of S , and only the upper part of $H(S)$ needs to be constructed. Because of the special geometric position of the points in S ,

at each step of the gift-wrapping algorithm we need to take into account only the subset of input points, and also we may eliminate some points of S from any further consideration at each step of the algorithm.

Algorithm 2 (for computing digital convex fuzzy hull):

step 1: Initialize

$\mathcal{E} = \emptyset$ (\mathcal{E} is the set of extremal points),

$\mathcal{L} = \emptyset$ (\mathcal{L} is the list of unprocessed facets, and the unprocessed edges of the facets),

$\mathcal{A}_H = \emptyset$ (\mathcal{A}_H is the set of internal - nonextremal points).

step 2: Let $\tilde{A}_1 \in \tilde{\mathcal{A}}$ be the point with maximum z coordinate, and let $\tilde{A}_2 \in \tilde{\mathcal{A}}$ be such that the angle between a vector parallel to the line $p(\tilde{A}_1, \tilde{A}_2)$, and vector $\vec{k} = (0, 0, 1)$, is minimal. Next, let $\sigma_0 = \sigma_0(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ be the plane determined by non-collinear points \tilde{A}_1, \tilde{A}_2 , and \tilde{A}_3 from $\tilde{\mathcal{A}}$, such that the angle between the normal vector to σ_0 , and the vector \vec{k} is minimal. (We rename the points, if necessary.) Note that all points of $\tilde{\mathcal{A}}$ are either in the plane σ_0 , or below it. Let us find all points \tilde{A}_k from $\tilde{\mathcal{A}}$, which belong to σ_0 . Let us denote by \mathcal{S}_0 the convex polygon, which is the convex hull of the set of such points, and by $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_r$, the set of vertices of \mathcal{S}_0 . (This can be done by finding the projection of such points on the Oxy plane, and using some two dimensional convex hull algorithm.)

Let us remove the points E_1, E_2, \dots, E_r , which correspond to vertices $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_r$ of \mathcal{S}_0 , from \mathcal{A} into the set \mathcal{E} of extremal points. Each time we remove some points A_k from \mathcal{A} , we remove simultaneously the corresponding points \tilde{A}_k from the set $\tilde{\mathcal{A}}$. A convex polygon \mathcal{S}_0 is a facet of the convex hull $H(S)$. We put \mathcal{S}_0 in the list \mathcal{L} of facets of $H(S)$, together with the list $(\tilde{E}_1\tilde{E}_2, \tilde{E}_2\tilde{E}_3, \dots, \tilde{E}_{r-1}\tilde{E}_r, \tilde{E}_r\tilde{E}_1)$ of edges of \mathcal{S}_0 , ordered counter-clockwise. We remove from \mathcal{A} the points A_k , which correspond to the remaining (non-extremal) points $\tilde{A}_k \in \sigma_0$, and put them in the set \mathcal{A}_H . We find points $A_l \in \text{pr}(\mathcal{S}_0)$, for which $\tilde{A}_l(x_l, y_l, \mu_l) \notin \sigma_0$. We remove the corresponding points A_l from \mathcal{A} and we put them in \mathcal{A}_H with new membership value $\mu(A_l)$ so that $(x_l, y_l, \mu(A_l)) \in \sigma_0$.

step 3: Let $\tilde{E}_1\tilde{E}_2$ be one edge of the polygon \mathcal{S}_0 . Let us find a vertical plane through edge $\tilde{E}_1\tilde{E}_2$, and let $\pi_{\tilde{E}_1\tilde{E}_2}$ be the halfspace determined by that plane, which does not contain \mathcal{S}_0 . Let \tilde{A}_k , be the point from the set $\tilde{\mathcal{A}}$, such that \tilde{A}_k belongs to $\pi_{\tilde{E}_1\tilde{E}_2}$, and that the angle between the normal vector to the plane $\sigma_1 = r(\tilde{E}_1, \tilde{E}_2, \tilde{A}_k)$ and vector $\vec{k} = (0, 0, 1)$ is minimal. Let us find the convex hull \mathcal{S}_1 of the set of points of $\mathcal{B} = \tilde{\mathcal{A}} \cap \pi_{\tilde{E}_1\tilde{E}_2} \cap \sigma_1$. We remove the projections

of the vertexes of \mathcal{S}_1 from \mathcal{A} , and we put them in \mathcal{E} . We remove from \mathcal{A} the points A_k , which correspond to the remaining (non-extremal) points \tilde{A}_k of \mathcal{S}_1 , and put them in the set \mathcal{A}_H . We find points $A_l \in \text{pr}(\mathcal{S}_1)$, for which $\tilde{A}_l(x_l, y_l, \mu_l) \notin \sigma_1$. We remove the corresponding points A_l from \mathcal{A} and we put them in \mathcal{A}_H with new membership value $\mu(A_l)$ so that $(x_l, y_l, \mu(A_l)) \in \sigma_1$.

We put the convex polygon \mathcal{S}_1 in the list \mathcal{L} , together with the list of its unprocessed edges ordered counter-clockwise. We remove the edge $\tilde{E}_1\tilde{E}_2$ from \mathcal{L} .

We repeat step 3 with each unprocessed edge of \mathcal{S}_0 .

We iteratively process all facets which we retrieve from the list \mathcal{L} , by iteratively processing the unprocessed edges of such facets.

The iteration process stops when $\mathcal{A} = \emptyset$, or $\mathcal{L} = \emptyset$.

$$\mathcal{A}_{FH} = \mathcal{E} \cup \mathcal{A}_H.$$

The end.

Let us denote by $\delta^{-1}(\mathcal{A}_{FH})$ the digital fuzzy set which corresponds to a fuzzy set \mathcal{A}_{FH} , i.e., the digital fuzzy set whose support consists of a set of pixels $P = \delta^{-1}(p)$, where $p \in \text{supp}(\mathcal{A}_{FH})$, with the same membership function ($\mu(P) = \mu(p)$).

Theorem 4 *The digital fuzzy set $\delta^{-1}(\mathcal{A}_{FH})$, derived by the previous algorithm, is the digital convex fuzzy hull $\mathcal{H}(A)$ of set A .*

Proof. By the preprocessing step of the Algorithm,

$$H'_{DL}(\text{supp}(A)) = \mathcal{A} = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\},$$

so the equation 3 of Definition 13 is satisfied. The convex hull $H(\mathcal{A}^0 \cup \tilde{\mathcal{A}}) = H(\mathcal{A}^0 \cup \tilde{\mathcal{A}}_{FH})$ (where $\tilde{\mathcal{A}}_{FH} = \{(x_k, y_k, \mu_k) | (x_k, y_k) \in \mathcal{A}_{FH}\}$) obtained by our modification of the gift-wrapping method is convex, so the fuzzy hypograph of \mathcal{A}_{FH} is convex. By Theorem 2, fuzzy set \mathcal{A}_{FH} (with membership function μ determined by the algorithm) is also convex, i.e., condition 4 of Definition 13 is satisfied. By construction, $\delta^{-1}(\mathcal{A}_{FH})$ is the smallest fuzzy set satisfying the conditions of Definition 13.

Thus, $\mathcal{QH}(A) = \delta^{-1}(\mathcal{A}_{FH})$. \square

4 Conclusion and Future Work

We introduced the notions of a digital quasiconvex fuzzy hull $\mathcal{QH}(A)$, and of a digital convex fuzzy hull $\mathcal{H}(A)$, of a given digital fuzzy set A . $\mathcal{QH}(A)$ and $\mathcal{H}(A)$ are both defined on a set of pixels, and they both have the same support $H_{DL}(A)$. $\mathcal{QH}(A)$ is a quasiconvex

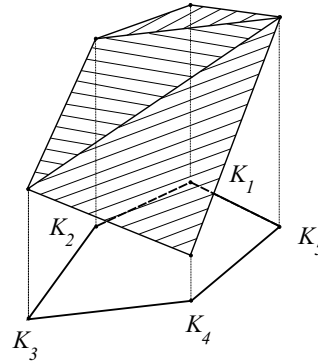


Figure 8: Illustration of the gift-wrapping algorithm.

set, while $\mathcal{H}(A)$ is a convex set. We proposed an algorithm for the construction of both digital fuzzy hulls.

We intend to extend this work by considering other notions of convexity of digital fuzzy sets, such as α -convexity.

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