Fuzziness - Representation of Dynamic Changes ?

Witold Kosiński
Polish-Japanese Institute
of Information Technology
Research Center
ul. Koszykowa 86, 02-008 Warsaw, PL
wkos@pjwstk.edu.pl

Piotr Prokopowicz
Kazimierz Wielki University in Bydgoszcz
Institute of Environmental Mechanics
and Applied Computer Science
ul. Chodkiewicza 30, 85-064 Bydgoszcz, PL
piotrekp@ukw.edu.pl

Abstract

The paper brings a discussion about the source of the inaccuracy in observations of objects and demonstrates that the essential reason of the lack of precision is changeability, and the more changeability, i.e. more dynamics, can be experienced the more inaccurate, more fuzzy judges can be. The space of ordered fuzzy numbers (OFN), the new model of fuzzy numbers that make possible to deal with fuzzy inputs quantitatively, exactly in the same way as with real numbers, is shortly presented. The new model possesses a set of properties which are in accordance with the influence of changeability on the increase of the inaccuracy in observations of the environment. The use of OFN is getting rid of the main problem in a classical fuzzy numbers - an unbounded increase in inaccuracies with next calculations. Moreover, new interpretation can be treated as an extend of classic proposals so there is no need to abandon existing ideas to deal with the new model of fuzzy numbers..

Keywords: Source of uncertainty, Fuzzy number, Ordered fuzzy number, Interpretation of fuzzy numbers, Algebraic operations.

1 Introduction

Fuzzy concept have been introduced in order to model such vague terms as observed values of some physical or economical terms, like pressure values or stock market rates, that can be inaccurate, can be noisy or can be difficult to measure with an appropriate precision because of technical reasons. In our daily life there are many cases that observations of objects in a population are fuzzy.

Discussion about the source of that inaccuracy is an aim of this publication. Authors want here to demonstrate that the essential reason of lack of precision in world's observing is changeability and the more changeability can be experienced the more inaccurate (more fuzzy) judges can be. Authors are introducing the new model of fuzzy numbers [9],[10],[11] defined by themselves together with Dominik Ślęzaka. The new model possesses a set of properties which are in accordance with the influence of the changeability on the increase of the inaccuracy in observations of the environment. Interesting thing is that the new interpretations supplied by the new model can be treated as an extend of classic proposals so we do not need abandon existing ideas to deal with new ones. Beside a little bit of different interpretation, the new model of fuzzy numbers has a lot of useful mathematical properties, in the particular we are getting rid of the main problem in a classical fuzzy numbers - the unbounded increase of inaccuracies with next calculations. Moreover, thanks to the new attempt we can define new methods - based on the arithmetic of ordered fuzzy numbers - of processing information in processes dealing with fuzzy control [14],[15].

2 Changes as source of uncertainty

We can ask a question: *which kind of person is an expert?* A possible answer seems obvious - he/she is a specialist in solving some kind of problems which can be described by a set of parameters. Those parameters should be at least in a number of few variables, in other way he/she could solve only one unique problem and it could be difficult say about him/her - the expert. So we can say: *more solvable problems with more variables and with wider ranges of values the person can describe, the better expert he/she is.* In fact if the one is a high class expert then he/she probably does not call a changeable his/her common situations, however another non-expert will see many changes around on the expert place. Point is the *changes* in this article should be treated relatively, not only in straight meaning of word changes. Now we can analyze some examples.

Let us imagine a situation, in which Mr. D. - an expert in assessing the distance - came on picnic out of the city. Let us establish, that while resting on the grass he has a good view on the nearby valley, in which a supermarket was built and many people are arriving for shopping. There is a crossroad with a quite

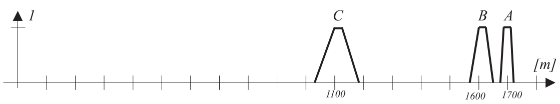

Figure 1: Assessments of Mr.D.

busy way at the end of the valley, and the majority of customers must stop there before living the valley. Observing cars which are starting from the parking lot Mr. D. can very accurate (the more accurate, the better expert he is) assess how long the road distance they must pass before reaching the crossroad. Now let us suppose a fuzzy number A (Fig.1) represents his assessment.

However, Mr. D.'s assessment of the distance from the place a given car stars to the crossroad becomes less precise when the car is in motion. The cause is the dynamics of the observed car. Faster the car drives, the less certain assessment is. Now let us allow fuzzy numbers B and C to represent the opinion about the distance in the tenth and twentieth seconds of observation of the moving car. It is of course pre-arranged script of assessments, however, intuitively the majority of people will confirm the fact that "fuzziness" of consecutive numbers should increase, at least till the moment of reaching the monotonous speed of the observed phenomenon.

Let us elaborate the example. Let us suppose the Mr. D. is great enjoying the picnic in the company of his friends and Mr. V. - an expert in assessing a velocity of moving objects. Mr. V. is observing the valley and he is able to describe with a high precision the speed of monotonously moving lorry, and this represents a fuzzy number S. However, the certainty of his assessment is less when he is trying to establish a velocity of a motorbike which is overtaking the lorry; in this case he gives a fuzzy number T. Moreover, if the motorbike all the time is speeding up and then slowing down overtaking next vehicles on the road, the precision of the assessment of Mr. V. is smaller and smaller. This represents a fuzzy number U.

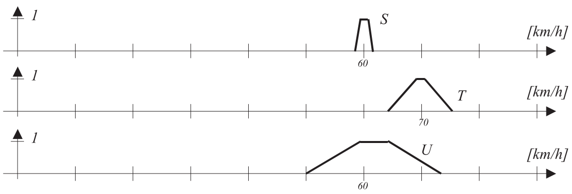

Figure 2: Assessments of Mr.V.

Alike as in the case of Mr. D. in the moment when well identified situations (i.e. monotonous speed of the object) begin to change, the uncertainty of assessments of Mr. V. is growing.

One can look for different examples showing the more changeable situations in which the uncertainty (as well as fuzziness) of assessments is growing. They could concern very different situations e.g. the teacher does not have a problem with assessing the pupil if his progress for the entire semester is monotonously growing, however, when the pupil once writes a very good work, another time a very crummy one, so in the course of the semester, the justice assessment is difficult and doubts can easily appear. Another example refers to prices of shares on stock exchange. When changes are very dynamic even the best experts will have some difficulties in assessing and a large portion of the uncertainty of their predictions will appear.

Perhaps one should not regard dynamics of changes in observed parameters as the only source of uncertainties, however, we can see that it obviously influences the precision of experts' assessments. One can give some reasons for linking uncertainty, and inaccuracy with dynamics of changes. Certainly one of them is rather the imprecise term: *now*. Since it is very hard for people to determine the exact moment of carrying the assessment out. Very notion *now* is a very inaccurate term. Sometimes it is indicating the given second, other time an hour and yet another time can mean even years (especially at economic assessments). Every *change* has a specific property which is a direction. In next part of this publication a new model of fuzzy numbers will be introduced - the *ordered fuzzy numbers*. They form a good tool to represent the imprecision understood exactly as a result of changes observed in values of parameters.

3 Critiques of convex fuzzy numbers

As long as one works with fuzzy numbers that possess continuous membership functions the two procedures: the extension principle and the α-cut and interval arithmetic method give the same results (cf. [1]) as far as their arithmetic. However, approximations of fuzzy functions and operations are needed if one wants to follow the extension principle and stay within (L, R)–numbers. It leads to some drawbacks as well as to unexpected and uncontrollable results of repeatedly applied operations [16].

Classical fuzzy numbers are very special fuzzy sets defined on the universe of all real numbers. If for a fuzzy set A defined on reals $\mathbf{R}$, we call

- the core of A as the (classical) set of those $x \in \mathbf{R}$ for which its membership function $\mu_A(x) = 1$, and
- the α-cut of A as a (classical) set $A[\alpha] = \{x \in \mathbf{R} : \mu_A(x) \geq \alpha\}$, for each $\alpha \in [0, 1]$, and
- the support of A as the (classical) set $\text{supp } A = \{x \in \mathbf{R} : \ \mu_A(x) > 0\}$,

then we are ready to define the so-called convex fuzzy numbers as those fuzzy sets A's on $\mathbf{R}$ that satisfy three conditions (compare [1],[2],[3],[13],[16]): a) the core of a fuzzy number A is nonempty, b) α-cuts of A are closed, bounded intervals, and c) supp$A$ is bounded. Since no assumption about continuity of the membership function μ_A of the fuzzy number has been made all crisp numbers are fuzzy numbers, as well.

The results of multiply operations on the convex fuzzy numbers are leading to the large grow of the fuzziness, and depend on the order of operations since the distributive law, which involves the interaction of addition and multiplication, does not hold there.

In this paper we will repeat our main arguments presented in the series of papers [7],[8],[9],[10],[11],[14],[15], that lead to a generalization of the classical concept of fuzzy numbers and then to new definition of ordered fuzzy numbers and their algebra which brings an evolutionary algorithm making possible its determination.

4 Inverse representation of membership functions

Our main observation made in [8] was: a kind of quasi-invertibility of membership functions is crucial and one has to define arithmetic operations on their inverse parts to be in agreement with operations on the crisp real numbers. Consequently, assuming this, the invertibility of membership functions of convex fuzzy number A makes it possible to define two functions a_1, a_2 on $[0,1]$ that give lower and upper bounds of each α-cut of the membership function μ_A of the number A

$$A[\alpha] := \{x \in \mathbf{R} : \mu_A(x) \geq \alpha\} = [a_1(\alpha), a_2(\alpha)], \quad (1)$$

where boundary points are given for each $\alpha \in [0,1]$ by

$$a_1(\alpha) = \mu_A|_{incr}^{-1}(\alpha) \text{ and } a_2(\alpha) = \mu_A|_{decr}^{-1}(\alpha) \ . \quad (2)$$

In (2) the symbol $\mu_A|_{incr}^{-1}$ denotes the inverse function of the increasing part of the membership function $\mu_A|_{incr}$, the other symbol refers to the decreasing part $\mu_A|_{decr}$ of μ. Then we can see that the membership function μ_A of A is completely defined [1] by two functions $a_1 : [0,1] \to \mathbf{R}$ and $a_2 : [0,1] \to \mathbf{R}$. In terms of them arithmetic operations on the set of fuzzy numbers are defined [1],[2],[13],. For example: if A and B are two (convex) fuzzy numbers with the corresponding functions a_1, a_2 and b_1, b_2 for A and B, respectively, then in terms of their α-cuts the result $C = A + B$ of addition is defined as follows:

$$C[\alpha] = A[\alpha] + B[\alpha], \ \alpha \in [0,1] \ , \quad (3)$$

$$C[\alpha] = [a_1(\alpha) + b_1(\alpha), \ a_2(\alpha) + b_2(\alpha)].$$

For subtraction, however, according to the interval arithmetic [5] the difference $D = A - B$ is defined

$$D[\alpha] = [a_1(\alpha) - b_2(\alpha), \ a_2(\alpha) - b_1(\alpha)], \ \alpha \in [0,1] \ . \quad (4)$$

Notice, that in subtraction of the same fuzzy number A, i.e. for $C = A - A$, we get $C[\alpha] = [a_1(\alpha) - a_2(\alpha), a_2(\alpha) - a_1(\alpha)]$ which represents non-crisp, fuzzy zero, unless $a_1(\alpha) = a_2(\alpha)$ for each α.

However, when the classical denotation for independent and dependent variables of the membership functions, namely x and y is used, and we look once more at (1)-(2), and if we put $y = \alpha$ and use x for the denotation of values of the functions a_1 and a_2, then we will get for two "wings" of the graph of A possible representations:

$$x = a_1(y) \text{ and } x = a_2(y) \ , \ y \in [0,1] \ . \quad (5)$$

In what follows we will use the approach (5) in the representation of so-called ordered fuzzy numbers identified with pairs of continuous functions of the interval $[0,1]$.

5 Ordered fuzzy numbers

In the series of papers [7],[6],[9],[10],[11],[12], [14],[15] we have introduced and then developed main concepts of the space of ordered fuzzy numbers. In our approach the concept of membership functions has been weakened by requiring a mere *membership relation*. Following our observations made in section 4 a fuzzy number A will be identified with the pair of functions a_1 and a_2 (cf. (1) - (2)) defined on the interval $[0,1]$, i.e.

Definition 1. *By an ordered fuzzy number A we mean an ordered pair of two continuous functions*

$$A = (x_{up}, x_{down})$$

called the up-branch and the down-branch, respectively, both defined on the closed interval $[0,1]$ with values in $\mathbf{R}$.

Notice, however, that in our definition we do not require that two continuous functions are inverse functions of some membership function. Moreover, in general a membership function corresponding to A may not exist.

The continuity of both parts implies their images are bounded intervals, say UP and $DOWN$, respectively (Fig. 2a)). If we used the symbols $UP = [l_A, 1_A^-]$ and $DOWN = [1_A^+, p_A]$ to mark boundaries and add the third interval $CONST = [1_A^-, 1_A^+]$, then we can see that are in fact three subintervals appearing in splitting the support of each convex fuzzy number, discussed above. Notice that in general neither $l_A \leq 1_A^-$ nor $1_A^+ \leq p_A$ must hold (i.e. $x_{up}(1)$ does not need to be less than $x_{down}(1)$). In this way we can reach improper intervals, which have been already discussed in the framework of the extended interval arithmetic by Kaucher in [4] and called by

[1] The boundary points of the core of A, i.e. the set on which the membership function attains value one, are defined by two values $a_1(1)$ and $a_2(1)$.

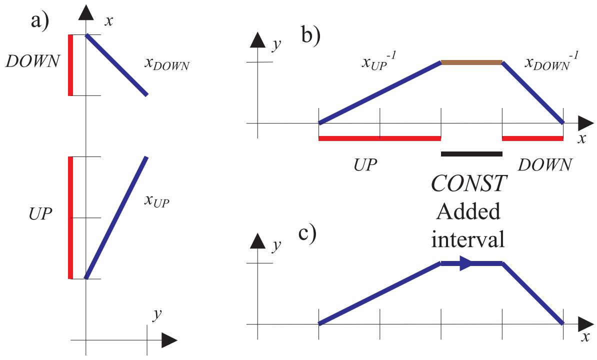

Figure 3: a) Ordered fuzzy number, b) Ordered fuzzy number with membership function, c) Arrow denotes the order of inverted functions and the orientation.

him directed intervals, i.e. such $[n,m]$ where n may be greater than m.

In general, the functions x_{up}, x_{down} need not to be invertible, if we assume, however, that they are monotonous: x_{up} is increasing, and x_{down} is decreasing, and such that $x_{up} \leq x_{down}$ (pointwise), we may define the membership function $\mu(x) = x_{up}^{-1}(x)$, if $x \in [x_{up}(0), x_{up}(1)] = [l_A, 1_A^-]$, and $\mu(x) = x_{down}^{-1}(x)$, if $x \in [x_{down}(1), x_{down}(0)] = [1_A^+, p_A]$ and $\mu(x) = 1$ when $x \in [1_A^-, 1_A^+]$.

In this way we have obtained the membership function $\mu(x), x \in \mathbf{R}$. When the functions x_{up} and/or x_{down} are not invertible or the second condition is not satisfied then the membership curve (or relation) can be defined, composed of the graphs of x_{up} and x_{down} and the line $y = 1$ over the core $\{x \in [x_{up}(1), x_{down}(1)]\}$.

It is worthwhile to point out that a class of ordered fuzzy numbers (OFNs) represents the whole class of convex fuzzy numbers ([1],[2],[3],[13],[16]) with continuous membership functions. In Fig. 3 c) to the ordered pair of two continuous functions (here just two affine functions) x_{up} and gx_{down} corresponds a membership function of a convex fuzzy number with an extra arrow which denotes the orientation of the closed curve formed below. This arrow shows that we are dealing with the ordered pair of functions.

A pair of continuous functions (x_{down}, x_{up}) determines different ordered fuzzy number than the pair (x_{up}, x_{down}). Graphically the plotes of (x_{up}, x_{down}) and (x_{down}, x_{up}) do not differ, however, the corresponding curves determine two different ordered fuzzy numbers: they differ by the *orientation* which we have denoted in Fig.3c by an arrow.

The original definition of OFNs from [9] has been recently generalized in [12].

Now, in the most natural way, the operation of addition between two pairs of such functions has been defined as the pairwise addition of their elements. This is exactly the same as the operation defined in Sec. 4 on α-cuts of A and B, cf. (3). As long as we are adding ordered fuzzy numbers which possess their classical counterparts in the form of trapezoidal type membership functions, and moreover, are of the same orientation, the results of addition are in agrement with the α-cut and interval arithmetic. However, this does not hold, in general, if the numbers have opposite orientations, for the result of addition may lead to improper intervals as far as some α-cuts are concerned. In this way we are close to the Kaucher arithmetic [4] with improper intervals.

Definition 2. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called ordered fuzzy numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A \div B$ are defined by formula*

$$f_C(y) = f_A(y) \star f_B(y) \ , g_C(y) = g_A(y) \star g_B(y) \quad (6)$$

where "$\star$" works for "$+$", "$-$", "$\cdot$", and "$\div$", respectively, and where $A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero.

As it was already noticed in the previous section the subtraction of B is the same as addition of the opposite of B, i.e. the number $(-1) \cdot B$.

6 Ordered fuzzy numbers around us

Model of ordered fuzzy numbers provides some interesting properties ([7],[9],[14],[15]), which open new areas for calculating and pro-

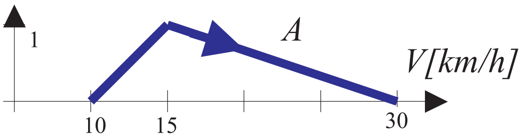

Figure 4: An example of the OFNs describing "slow in speed-up process".

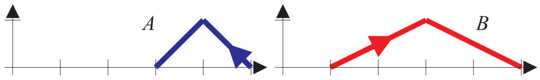

Figure 5: An example of the OFNs that describes income in two units of a financial company.

cessing vague information. Very important for every idea is how it refers to the real life. The interpretation of OFNs together with their orientation will be presented here.

The common use of fuzzy numbers is the presentation and the operation on imprecise data. In general, that is also a source of the idea of all fuzzy sets. Interpretation of the ordered fuzzy numbers is compatible with the general idea of the fuzzy sets. However, there exists a new property - the orientation. By using OFNs we can describe any imprecise value in the real-life processes. The parts up-branch and down-branch of OFN can be related to an opinion of an expert about dynamic changes of the analyzed value. The up-branch describes the behaviour of the value before the very moment when the opinion was made, and the down-branch describes value in afterwards. In that way we expand existing interpretation of fuzzy numbers. We can still use OFNs in the way as usual when we ignore the orientation, but we can also use the orientation to put more complex information about the evaluation made by OFNs.

Let us look at the example in which we have an imprecise opinion "slow" about the speed of a vehicle as OFN A (see Fig. 4). We can ignore the orientation and use this OFN as fuzzy data by saying speed 15 is surely slow and speeds 13 and 20 are slow in degree little more than 50%.We can also take into consideration the orientation of OFN and can say: it is "slow in the speed-up process".

Let us look for another example from the economy and consider a financial company, which has two units A and B. Expert made opinion about the income of both units. For A he said: "income is stated on level 4 millions and this is a downward trend". For B he said: "income is stated on level 3 millions and this is a upward trend". He described incomings of both units by two OFNs (Fig. 5) A and B. By using OFNs the expert can describe not only the value and the trend but also the escalation of that trend.

We have two OFNs where "wide" of branches (up and down) are different. Number B is more "wide" than A. What does it mean? We can find answer if we make more deep (but simply) analysis. If the expert has made up-branch of A from 5 to 4 millions then he considers possible range of changes as 1 million. Up-branch of B was made from 1 to 3 millions so he considers range of changes as 2 millions. To sum up, we understand the number B as an information about a process which us more dynamic than A. Another thing is the direction that shows that A is the decreasing process and B is the increasing one.

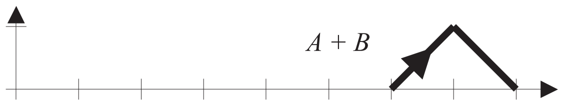

Figure 6: The total income of the company as the sum of A and B.

In real life we could expect total income of analyzed company about 7 millions. Additional, if the increasing process of B was more dynamic than decreasing of A then we expect in total also increasing process, however less dynamic than for B. If we use OFN model and add numbers A and B according to (6) then we get expected results (Fig. 6).

7 Conclusions

The ordered fuzzy numbers form a tool for describing and processing vague information. They expand existing ideas. Their "good" algebra opens new areas for calculations. Beside that, new property (orientation) and its interpretation presented in this paper can open new areas for using fuzzy numbers. Important fact (in author's opinion) is that thanks to OFNs we can join without complication classical field of fuzzy numbers with new ideas. We can use the OFNs instead the convex fuzzy numbers and if we need to use extended properties we can use them easily. One of directions of the future work with the OFNs are rules in the inference system for a fuzzy controller with new rules. The OFN can contain much more information than the classical fuzzy number - so why do not use it?

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