How to use controller with conditionally firing rules

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Abstract

In previous work, a fuzzy controller with conditionally firing rules was suggested as an advantageous alternative to Mamdani–Assilian controllers. Its features have been used without much understanding. Therefore we explicitly formulate the rules of its design in this tutorial which has not been available before.

Keywords: fuzzy controller, compositional rule of inference, firing rule, fuzzy relational equation, fuzzy hardware.

1 Introduction

A fuzzy controller with conditionally firing rules (CFR controller) was introduced in [10]. It is a generalization of the Mamdani–Assilian controller, using an inference rule that is not compositional. It was motivated by the effort to fit the behavior of the controller optimally to the meaning of the fuzzy rule base. Both theoretical and practical arguments support the expectation that the CFR controller allows a better performance without a change of the rule base. Thus it may improve the quality of control while keeping the order of computational complexity. It has been applied in several tasks [6, 8, 11, 13, 14], but the users implemented it sometimes in a simple manner, without much understanding its options and capabilities. Therefore we found it necessary to write a "user's manual" for this type of controller which we present here.

2 Theoretical analysis of fuzzy controllers

For basics on fuzzy controllers, we refer to [4, 5]. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the input and the output space of a controller, respectively. They are supposed to be convex subsets of finite-dimensional real vector spaces. For a set $\mathcal{Z}$, we denote by $\mathcal{F}(\mathcal{Z})$ the set of all its fuzzy subsets. Throughout this paper, we identify fuzzy sets with their membership functions. The expert's knowledge may be expressed by a base Θ of rules of the form

$$\textbf{if } x \in A_i \textbf{ then } y \in C_i \,,$$

where $A_i \in \mathcal{F}(\mathcal{X})$ are *antecedents* and $C_i \in \mathcal{F}(\mathcal{Y})$ are *consequents*, $i \in \{1, \ldots, n\}$ (see [4]). Following [15], the knowledge from the rule base Θ can be represented by a fuzzy relation $R_\Theta \in \mathcal{F}(\mathcal{X} \times \mathcal{Y})$. The Mamdani–Assilian controller [7] uses R_Θ in the form

$$R_\Theta(x, y) = \max_{i \le n} T(A_i(x), C_i(y)) \,, \tag{1}$$

where T is a fixed t-norm modelling a fuzzy conjunction [5]. Then we apply a *compositional rule of inference* which assigns to a fuzzy input $X \in \mathcal{F}(\mathcal{X})$ a fuzzy output $Y \in \mathcal{F}(\mathcal{Y})$ by $Y = X \underset{T}{\circ} R_\Theta$, i.e.,

$$Y(y) = \sup_{x \in \mathcal{X}} T(X(x), R_\Theta(x, y)) \,. \tag{2}$$

We denote the induced mapping by $\Phi_\Theta^{\mathrm{MA}} \colon \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$. Combining formulas (1), (2), it can be expressed as

$$\begin{aligned} Y(y) &= \Phi_\Theta^{\mathrm{MA}}(X)(y) \\ &= \sup_{x \in \mathcal{X}} T\Big(X(x), \max_{i \le n} T(A_i(x), C_i(y))\Big) \\ &= \max_{i \le n} T\Big(\sup_{x \in \mathcal{X}} T(X(x), A_i(x)), C_i(y)\Big) \,. \end{aligned}$$

The latter form allows for an effective calculation using the values

$$\mathcal{D}_T(X, A_i) = \sup_{x \in \mathcal{X}} T(X(x), A_i(x)) \,, \tag{3}$$

called *degrees of overlapping* of X, A_i. The value $\mathcal{D}_T(X, A_i)$ is the height of the fuzzy intersection $X \underset{T}{\cap} A_i$. The output is

$$Y(y) = \max_{i \le n} T\Big(\mathcal{D}_T(X, A_i), C_i(y)\Big) \,. \tag{4}$$

The use of degrees of overlapping allows to avoid nested cycles over the spaces $\mathcal{X}$, $\mathcal{Y}$. Notice that the simplification is possible due to the associativity of T; it requires to use the same t-norm T in (1) and (2).

3 Requirements on fuzzy controllers

To formulate mathematically the desirable properties of a fuzzy controller, we have to introduce several basic notions and notations from fuzzy set theory. A fuzzy set is called *normal* if its membership function attains the value 1 at (at least) some point. The *support* of a fuzzy set $A \in \mathcal{F}(\mathcal{Z})$ is

$$\operatorname{Supp} A = \{z \in \mathcal{Z} : A(z) > 0\} .$$

For $x \in \mathcal{Z}$, let $\chi_x \in \mathcal{F}(\mathcal{Z})$ denote a singleton (crisp value), i.e.,

$$\chi_x(z) = \begin{cases} 1 & \text{if } z = x\,, \\ 0 & \text{if } z \neq x \end{cases}$$

for all $z \in \mathcal{Z}$. A fuzzy set is called *convex* if all its α-cuts are convex sets. For fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathcal{Z})$, their *convex hull* is the smallest (w.r.t. the pointwise ordering) convex fuzzy set $C \in \mathcal{F}(\mathcal{Z})$ satisfying $A_i(z) \leq C(z)$ for all $z \in \mathcal{Z}$, $i = 1, \ldots, n$.

Let $\Theta = (A_i, C_i)_{i=1}^n$ be a rule base. We want the control function, $\Phi_\Theta : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$, of a fuzzy controller to satisfy the following properties:

[I1] If the input coincides with the ith antecedent, then the output coincides with the ith consequent, i.e., $\Phi_\Theta(A_i) = C_i$.

[I2] For each normal input $X \in \mathcal{F}(\mathcal{X})$, the output $\Phi_\Theta(X)$ is not contained in all consequents, i.e., $\Phi_\Theta(X) \not\leq \min_{j \leq n} C_j$.

[I3] The output $\Phi_\Theta(X)$ belongs to the convex hull of consequents C_i of all rules such that $\operatorname{Supp} A_i \cap \operatorname{Supp} X \neq \emptyset$ (i.e., all firing rules).

[I4] Let $X = \chi_x$ for $x \in \mathcal{X}$ such that $A_i(x) = 1$ (for a fixed i). Then $\Phi_\Theta(X) = C_i$.

Condition [I2] avoids non-significant outputs bearing no information about the input. It compares the output with the fuzzy intersection of all consequents. Usually this intersection is empty, then [I2] simply means that the output is non-empty. Our formulation gives the respective meaning also in the (rare) case when the intersection of all consequents in non-empty.

Condition [I3] is a very weak requirement meaning that the output is obtained by *interpolation* of the consequents.

Condition [I4] concerns the (not so unusual) case when there is a totally firing rule for a crisp input. Then the output should be the corresponding consequent. This means that having a well-fitting rule, we do not need any other (which could only damage the correct output). As a consequence, we have to avoid a rule base in which this could occur simultaneously for more than one rule.

Condition [I1] deserves more attention. It is called *interaction* in [4]. It says that an antecedent as an input produces the corresponding consequent as an output. For a Mamdani–Assilian controller it corresponds to a system of fuzzy relational equations

$$C_i = A_i \underset{T}{\circ} R_\Theta\,, \qquad i \in \{1, \ldots, n\} . \tag{5}$$

Necessary and sufficient conditions for its solvability by a Mamdani–Assilian controller are given in [3] (cf. also [12]). To formulate them, we use the *residuum* (*residuated implication*) induced by T, i.e., the operation $I_T : [0,1] \times [0,1] \to [0,1]$ such that

$$I_T(a,b) = \sup\{c \in [0,1] : T(a,c) \leq b\} .$$

If T is continuous, we may take max instead of sup in the latter formula. The corresponding *biimplication* is

$$E_T(a,b) = \min\{I_T(a,b), I_T(b,a)\} .$$

For two consequents, C_i, C_j, we define their *degree of indistinguishability*, $\mathcal{E}_T(C_i, C_j)$, by

$$\mathcal{E}_T(C_i, C_j) = \inf_{z \in \mathcal{Z}} E_T(C_i(z), C_j(z)) .$$

Recall that the degree of indistinguishability $\mathcal{E}_T : \mathcal{F}(\mathcal{Y}) \times \mathcal{F}(\mathcal{Y}) \to [0,1]$ is a fuzzy equality relation (similarity) on $\mathcal{F}(\mathcal{Y})$ with respect to T.

Theorem 1 [3] *Let $\Theta = (A_i, C_i)_{i=1}^n$ be a rule base with $A_i(x_i) = 1$ for pairwise different elements $x_1, \ldots, x_n \in \mathcal{X}$. Then relation* (1) *satisfies the system* (5) *iff all $i, j \in \{1, \ldots, n\}$ satisfy*

$$\mathcal{D}_T(A_i, A_j) \leq \mathcal{E}_T(C_i, C_j) . \tag{6}$$

Condition (6) (and hence also [I1]) is violated in many fuzzy controllers. It depends on the choice of the t-norm T; choosing a smaller t-norm (e.g., the product instead of the minimum) could help to satisfy it. Nevertheless, this choice makes it more difficult to satisfy condition [I2]. Thus conditions [I1], [I2] are almost contradictory for a Mamdani–Assilian controller and typical shapes of membership functions. (See [2, 9, 10] for more detailed analysis of this phenomenon.)

4 CFR controller and its design

In [10], a controller is suggested which allows to satisfy the above requirements for rather general rule bases. It assumes the following properties of the rule base $\Theta = (A_i, C_i)_{i=1}^n$:

[C1] *"normality"*: each A_i is normal,

[C2] *"covering of antecedents"*: $\inf\limits_{x \in \mathcal{X}} \max\limits_{i \le n} A_i(x) > 0$,

[C3] *"significance of consequents"*: $\forall i \in \{1, \dots, n\}$: $C_i \not\le \min\limits_{j \ne i} C_j$.

Conditions [C1] and [C2] ensure that we have normal antecedents covering the whole input space and [C3] is quite a weak requirement on significance of consequents. See [2, 9] for more detailed analysis of these conditions.

The new controller is obtained from the Mamdani–Assilian controller by the following modifications:

Step 1. Change the membership degrees in the input space by an increasing bijection $\varrho\colon [0,1] \to [0,1]$. This changes the degrees of overlapping $\mathcal{D}_T(X, A_i)$ to

$$\mathcal{D}_T(X \circ \varrho, A_i \circ \varrho) = \sup_{x \in \mathcal{X}} T(\varrho(X(x)), \varrho(A_i(x))) \,. \quad (7)$$

To keep the numerical complexity of the changes of scale as low as possible, we may choose $\varrho(t) = t^r$, $r \in N$, or a piecewise linear function. Choosing $\varrho(t) \le t$, we may reduce the degrees of overlapping to an arbitrary small value (provided that they were < 1).

Step 1 can be skipped (choosing the identity for ϱ) unless the degrees of overlapping $\mathcal{D}_T(A_i \circ \varrho, A_j \circ \varrho)$ are too large for some i, j, $i \ne j$.

Step 2. Choose $c < 1$ such that

$$c \ge \mathcal{D}_T(A_i \circ \varrho, A_j \circ \varrho) \quad (8)$$

for all i, j, $i \ne j$. Change the membership degrees in the output space by an increasing bijection $\sigma\colon [0,1] \to [c, 1]$. To extend the inverse of σ to the whole interval $[0,1]$, we use its *pseudoinverse*

$$\sigma^{[-1]}(t) = \begin{cases} \sigma^{-1}(t) & \text{if } t \ge c\,, \\ 0 & \text{otherwise}\,. \end{cases} \quad (9)$$

We may choose for example $\sigma(t) = (1-c) \cdot t + c$.

Step 3. Replace the degree of overlapping in (4) with the *degree of conditional firing* of the ith rule defined as

$$\mathcal{C}_{T,i}(X) = \frac{\mathcal{D}_T(X \circ \varrho, A_i \circ \varrho)}{\max\limits_{j \le n} \mathcal{D}_T(X \circ \varrho, A_j \circ \varrho)} \,. \quad (10)$$

(In fact, the value $\mathcal{C}_{T,i}(X)$ depends not only on T and i, but also on ϱ and on all antecedents A_j, $j = 1, \dots, n$. For each normal input X, the values $\mathcal{C}_{T,i}(X)$, $i = 1, \dots, n$, belong to $[0,1]$ and at least one of them is 1.)

Thus the rules are activated according to their *relative* degree of overlapping with the input.

After all these three changes, the output becomes

$$\begin{aligned} Y(y) &= \Phi_\Theta^{\mathrm{CFR}}(X)(y) \\ &= \sigma^{[-1]}\Big(\max_{i \le n} T\big(\mathcal{C}_{T,i}(X), \sigma(C_i(y))\big)\Big) \,, \end{aligned} \quad (11)$$

where $\Phi_\Theta^{\mathrm{CFR}}\colon \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$ denotes the induced input-output correspondence of this modified controller, called a *controller with conditionally firing rules*, or briefly *CFR controller*.

We do not deal here with defuzzification. Our basic observations refer to fuzzy subsets of the input and output space and they are independent of the method of defuzzification.

Theorem 2 [10] *Let a rule base Θ satisfy conditions* [Con1–3]. *Then the CFR controller given by* (11) *satisfies properties* [I1–4] *(for any increasing bijection $\varrho\colon [0,1] \to [0,1]$, any $c < 1$ satisfying (8), and any increasing bijection $\sigma\colon [0,1] \to [c,1]$).*

Condition (8) represents disjointness of antecedents needed to distinguish between firing rules (not more than one rule may fire totally as this would cause a contradiction in the rule base).

The weakening of conditions on rule base enabled by a CFR controller is mainly due to the following fact:

Theorem 3 [10] *Let $\Theta = (A_i, C_i)_{i=1}^n$ be a rule base with $A_i(x_i) = 1$ for pairwise different elements $x_1, \dots, x_n \in \mathcal{X}$. Then the input–output correspondence Φ_Θ^{CFR} of a CFR controller (with mappings ϱ, σ described above) satisfies* [I1] *iff all $i, j \in \{1, \dots, n\}$ satisfy*

$$\mathcal{D}_T(A_i \circ \varrho, A_j \circ \varrho) \le \mathcal{E}_T(C_i \circ \sigma, C_j \circ \sigma) \,. \quad (12)$$

Under the conditions of Theorem 2, we may choose the mapping $\varrho\colon [0,1] \to [0,1]$ so that the left-hand side of (12) becomes an arbitrarily small positive number; the lower bound c of the range of the mapping $\sigma\colon [0,1] \to [c,1]$ gives a lower estimate of the right-hand side of (12), so inequality (12) can be satisfied.

5 Simplification of CFR controller for crisp inputs

In general, it is not possible to substitute a CFR controller in its full generality by a Mamdani–Assilian controller. Nevertheless, in the most important case of crisp inputs it admits such a modification [1].

A crisp input means that the values taken into account are represented by real numbers. Still they may be subject to some imprecision or uncertainty, but despite of this, the result is one value, not a fuzzy set of values. This is true for most measuring devices. In fact, it is very rare and difficult to measure a quantity in such a way that the result is given as a fuzzy set. Therefore a reduction to crisp inputs is assumed in most fuzzy controllers. Then fuzziness is reduced to the computations *inside* the controller, while it communicates with the world through crisp inputs and outputs. This approach is still general enough to be the basis of most applications of fuzzy control. In fact, this is what the current fuzzy hardware does; it handles only crisp inputs.

First we shall show how the controllers simplify if the input is crisp, i.e., $X = \chi_x$ for some $x \in \mathcal{X}$. Then the degrees of overlapping of X, A_i become

$$\mathcal{D}_T(X, A_i) = \sup_{x\in\mathcal{X}} T(X(x), A_i(x)) = A_i(x) \tag{13}$$

(independently of the choice of the t-norm T). Let us denote by $\varphi_\Theta^{\mathrm{MA}}(x)$ the output $\Phi_\Theta^{\mathrm{MA}}(X)$ of a Mamdani–Assilian controller corresponding to a crisp input $X = \chi_x$. Thus $\varphi_\Theta^{\mathrm{MA}}\colon \mathcal{X} \to \mathcal{F}(\mathcal{Y})$ is the mapping fully describing the action of the Mamdani–Assilian controller on crisp inputs. We obtain

$$Y(y) = \varphi_\Theta^{\mathrm{MA}}(x)(y) = \max_{i\le n} T(A_i(x), C_i(y))\,. \tag{14}$$

What remains is to defuzzify the fuzzy output Y.

In case of a CFR controller, we obtain

$$\begin{aligned}\mathcal{D}_T(X\circ\varrho, A_i\circ\varrho) &= \sup_{x\in\mathcal{X}} T(\varrho(X(x)), \varrho(A_i(x)))\\ &= \varrho(A_i(x))\,.\end{aligned}$$

(Notice that the membership degrees $X(x) \in \{0,1\}$ remain unchanged by ϱ.) The degrees of conditional firing are

$$\mathcal{C}_{T,i}(X) = \frac{\mathcal{D}_T(X\circ\varrho, A_i\circ\varrho)}{\max\limits_{j\le n}\mathcal{D}_T(X\circ\varrho, A_j\circ\varrho)} = \frac{\varrho(A_i(x))}{\max\limits_{j\le n}\varrho(A_j(x))}\,. \tag{15}$$

These values depend only on $A_i \circ \varrho$, $i \in \{1,\ldots,n\}$, which are known in advance (once the design of the controller is completed), and on x. Thus they can be represented internally by functions, similarly to A_i, $i \in \{1,\ldots,n\}$. Explicitly, we define fuzzy sets (modified antecedents) $\overline{A}_i \in \mathcal{F}(\mathcal{X})$, $i \in \{1,\ldots,n\}$, by

$$\overline{A}_i(x) = \frac{\varrho(A_i(x))}{\max\limits_{j\le n}\varrho(A_j(x))}\,. \tag{16}$$

Under the assumption [C2], they are well defined (the denominator is nonzero). Then the desired degree of conditional firing may be expressed as

$$\mathcal{C}_{T,i}(X) = \overline{A}_i(x)\,. \tag{17}$$

The transformation σ may be analogously represented by a modification of consequents. Explicitly, we define fuzzy sets (modified consequents) $\overline{C}_i \in \mathcal{F}(\mathcal{Y})$, $i \in \{1,\ldots,n\}$, by

$$\overline{C}_i(y) = \sigma(C_i(y))\,. \tag{18}$$

Let us denote by $\varphi_\Theta^{\mathrm{CFR}}(x)$ the output $\Phi_\Theta^{\mathrm{CFR}}(X)$ of a CFR controller corresponding to a crisp input $X = \chi_x$. Thus $\varphi_\Theta^{\mathrm{CFR}}\colon \mathcal{X} \to \mathcal{F}(\mathcal{Y})$ is the mapping fully describing the action of the CFR controller on crisp inputs. With the above modifications, we obtain

$$Y(y) = \varphi_\Theta^{\mathrm{CFR}}(x)(y) = \sigma^{[-1]}\Big(\max_{i\le n} T\big(\overline{A}_i(x), \overline{C}_i(y)\big)\Big)\,. \tag{19}$$

Except for the final transformation $\sigma^{[-1]}$, this formula is of the same form as formula (14) for the Mamdani–Assilian controller. More exactly, if $\overline{\Theta} = (\overline{A}_i, \overline{C}_i)_{i=1}^n$ is the modified rule base and $\varphi_{\overline{\Theta}}^{\mathrm{MA}}$ is the respective input-output correspondence, then

$$\varphi_\Theta^{\mathrm{CFR}}(x) = \varphi_{\overline{\Theta}}^{\mathrm{MA}}(x)\circ\sigma^{[-1]}\,.$$

This proves the following result:

Theorem 4 *Suppose that we have a controller with conditionally firing rules with a rule base Θ which performs a mapping $\varphi_\Theta^{CFR}\colon \mathcal{X} \to \mathcal{F}(\mathcal{Y})$. Let $\sigma\colon [0,1] \to [c,1]$ be the mapping used in (11). Then there is a Mamdani–Assilian controller with a rule base $\overline{\Theta}$ which performs a mapping $\varphi_{\overline{\Theta}}^{MA}\colon \mathcal{X} \to \mathcal{F}(\mathcal{Y})$ such that $\varphi_\Theta^{CFR}(x) = \varphi_{\overline{\Theta}}^{MA}(x)\circ\sigma^{[-1]}$ for all $x \in \mathcal{X}$, where $\sigma^{[-1]}$ is the mapping given by (9).*

This means that the implementation of a CFR controller with crisp inputs can be done using the standard Mamdani–Assilian controller and a block performing the transformation $\sigma^{[-1]}$. As a consequence, hardware for Mamdani–Assilian controllers can be used to implement a CFR controller.

6 Example

In this section we present examples demonstrating the use of the CFR controller. The first example shows how the controller computes the output value for a given set of antecedents, consequents, and an input value. The second example gives the output of the Mamdani–Assilian controller with the same antecedents, consequents, and rule base. A comparison

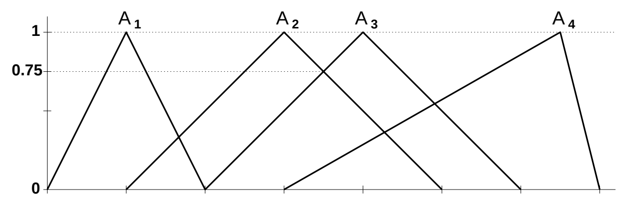

Figure 1: Example I: Fuzzy sets defining the antecedents.

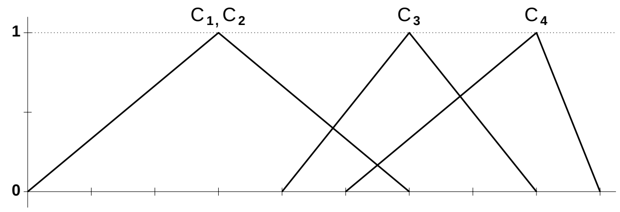

Figure 2: Example I: Fuzzy sets defining the consequents.

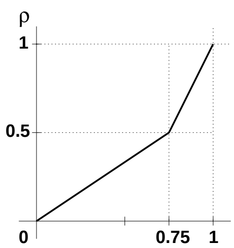

Figure 3: Example I: Graph of function ϱ.

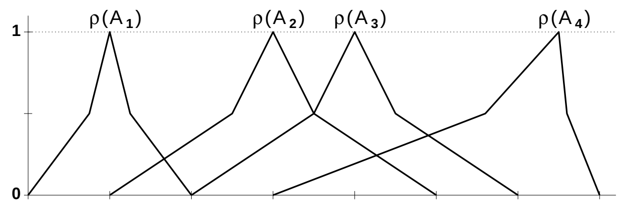

Figure 4: Example I: Antecedents modified by function ϱ.

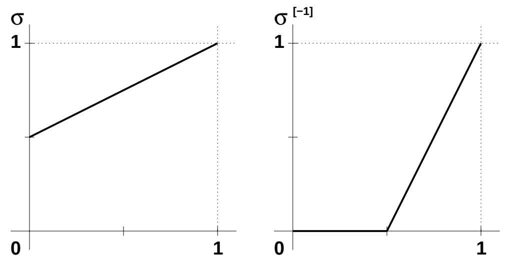

Figure 5: Example I: Graphs of functions σ and $\sigma^{[-1]}$.

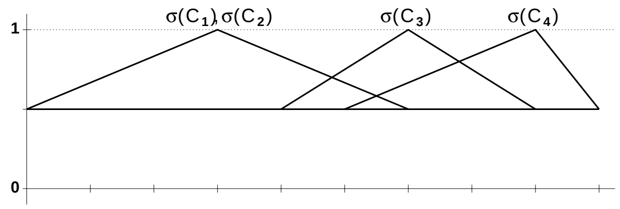

Figure 6: Example I and III: Consequents modified by function σ.

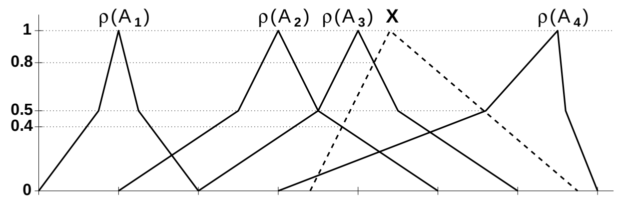

Figure 7: Example I: An input of the CFR controller represented by a fuzzy set X.

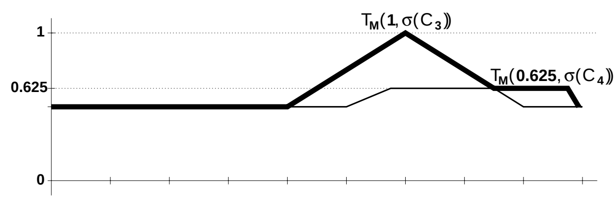

Figure 8: Example I: Consequents modified by function σ and combined with the degrees of conditional firing.

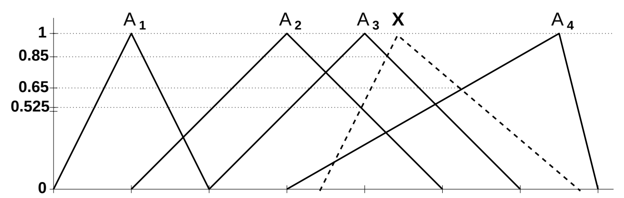

Figure 9: Example II: Antecedents, an input, and degrees of overlapping of the Mamdani–Assilian controller.

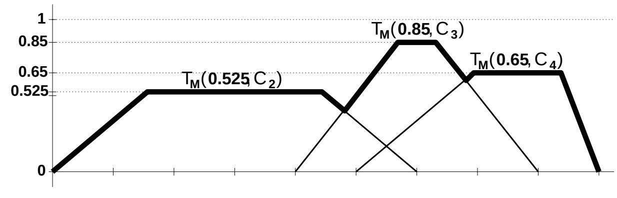

Figure 10: Example II: Consequents of the Mamdani–Assilian controller combined with the degrees of overlapping.

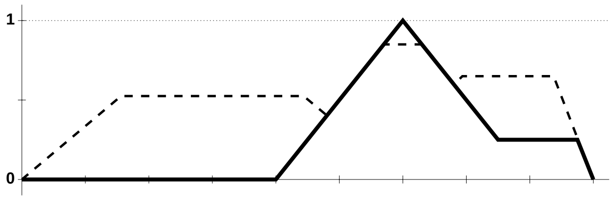

Figure 11: Example I and II: Resulting output fuzzy sets. The CFR controller is represented by the thick line; the Mamdani–Assilian controller is represented by the dashed line.

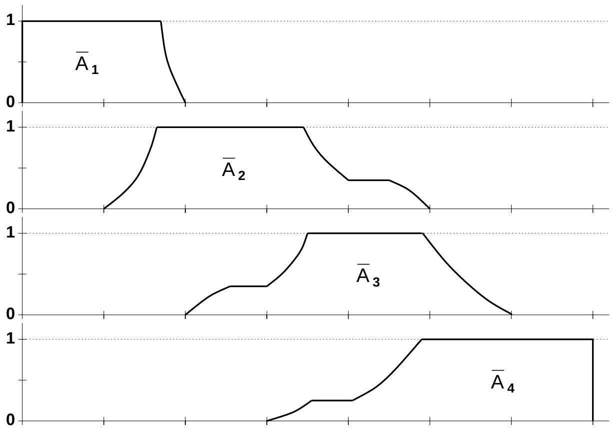

Figure 12: Example III: Antecedents of the derived Mamdani–Assilian controller.

of these two results is then made. The third example describes the transformation of a CFR controller into a Mamdani–Assilian controller for the case when the input can attain only crisp values. For simplicity, we demonstrate the principles in one dimension; the extension to more dimensions is easy.

6.1 Example I

We have a set of antecedents and consequents as given in Figure 1 and Figure 2. Further we have a set of fuzzy rules in the form:

1. **if** $x \in A_1$ **then** $y \in C_1$,
2. **if** $x \in A_2$ **then** $y \in C_2$,
3. **if** $x \in A_3$ **then** $y \in C_3$,
4. **if** $x \in A_4$ **then** $y \in C_4$,

To implement a CFR controller we have decided to use the standard (Gödel, minimum, ...) t-norm defined as $T_M(x, y) = \min\{x, y\}$. Now we need to find some suitable realizations for the functions ϱ and σ.

The function σ attains values from the interval $[c, 1]$ where $c \in (0, 1)$ is a number higher or equal to the highest degree of overlapping of the antecedents. Looking at Figure 1 we can see that the highest degree of overlapping is 0.75. Yet this value is too high; operating only in the interval $[0.75, 1]$ we could lose some precision. We may decide to have the constant c at most 0.5. For this, we choose the function ϱ as in Figure 3. Now the antecedents modified by ϱ look as shown in Figure 4, the highest degree of overlapping is 0.5, and we can choose $c = 0.5$. The functions σ and $\sigma^{[-1]}$ then may look as shown in Figure 5 and Figure 6 shows the consequents modified by σ.

Let us suppose that the input of the controller is a fuzzy set X from Figure 7. We are going to compute the output value.

First we determine the degrees of overlapping given by the fuzzy set X and the antecedents. The degrees are determined according to Figure 7 and they are listed in the fist row of Table 1. The highest degree of overlapping is 0.8 in our case. To compute the degrees of conditional firing, we divide each degree of overlapping by the highest degree, i.e. by 0.8.

	A_1	A_2	A_3	A_4
$\mathcal{D}(X)$	0	0.4	0.8	0.5
$\mathcal{C}(X)$	0	0.5	1	0.625

Table 1: Example I: Degrees of overlapping and degrees of conditional firing for the antecedents A_1, A_2, A_3, and A_4.

Having computed the degrees of conditional firing, we combine them with the consequents by the minimum t-norm, see Figure 8. (The first two consequents after the modification become less than 0.5 and they are omitted because they do not influence the output.) From these modified fuzzy sets the maximum is taken (shown in Figure 8 by a thick line) and this maximum is modified by the function $\sigma^{[-1]}$. This way, we obtain the resulting output fuzzy set shown in Figure 11 by a thick line. (It is subject to defuzzification which is not discussed here.)

Notice that although the degree of overlapping of the antecedent A_2, resp. A_4 was rather high, the influence of the corresponding consequent to the final result was none, resp. small.

6.2 Example II

For a comparison, let us now create a classical Mamdani–Assilian controller with the same antecedents, consequents, and rule base as in the first example. Let us suppose that the input of the controller is the same fuzzy set X (see Figure 9). We are going to compute the output value.

We determine the degrees of overlapping, given by the fuzzy set X and the antecedents, as can be seen in Figure 9; the degrees are listed in Table 2. We combine the degrees of overlapping with the consequents by the minimum t-norm, see Figure 10. (The first consequent after the modification becomes zero and it is omitted.) By taking maximum from these modified fuzzy sets (shown in Figure 10 by a thick line) we obtain the resulting output fuzzy set shown in Figure 11 by a dashed line.

	A_1	A_2	A_3	A_4
$\mathcal{D}(X)$	0	0.525	0.85	0.65

Table 2: Example II: Degrees of overlapping for the antecedents A_1, A_2, A_3, and A_4 in the case of the Mamdani–Assilian controller.

Now, in Figure 11, we can see the output fuzzy sets both for the CFR controller (thick line) and for the Mamdani–Assilian controller (dashed line) for the same antecedents, consequents, and rule base. As we can see the CFR controller gives a very specific result saying that the output is given mostly by the third consequent slightly influenced by the fourth consequent. On the contrary, the output of the corresponding Mamdani–Assilian controller is much more uncertain and blurred. The influence of all the three nonzero consequents is considerably high which makes the interpretation and the defuzzification more difficult.

6.3 Example III

In this example we are going to trasform a CFR controller to a Mamdani–Assilian controller according to Section 5. In general, it is not possible to substitute a CFR controller in its full generality by a Mamdani–Assilian controller. Nevertheless, in the most important case of crisp inputs it admits such a modification [1].

Note that this Mamdani–Assilian controller will work only with crisp input values. Let us take the same set of antecedents and consequents and the same rule base as in the first example. First we use (16) and modify the set of antecedents as it can be seen in Figure 12. Then, according to (18), we modify the set of consequents as it is shown in Figure 6. The rule base remains unchanged.

This way we have obtained new sets of antecedents and consequents. We can now apply the algorithm of the Mamdani–Assilian controller to the modified rule base. Then the mapping $\sigma^{[-1]}$ is applied to the output. Thus we obtain, for crisp inputs, the same behaviour as in the original CFR controller. It satisfies our requirements [I1–4] with respect to the *original* antecedents and consequents.

7 Discussion

Let us now discuss what Th. 2 says for the case of crisp input. (With the exception of [I1], the conclusions are valid for both the original formulation of a CFR controller and its reformulation described above—they are equivalent in this case.) Conditions [C1–4] remain unchanged. Property [I1] becomes meaningless, because we do not accept fuzzy inputs (the antecedents are usually fuzzy sets, not crisp singletons). Property [I2] simplifies, because crisp inputs are always normal (when considered as a special case of fuzzy sets). The new formulation can be:

[$\overline{\text{I2}}$] For each crisp input $x \in \mathcal{X}$, the output $\varphi_\Theta^{\text{CFR}}(x)$ is not contained in all consequents, i.e., $\varphi_\Theta^{\text{CFR}}(x) \not\leq \min_{j\leq n} C_j$.

Property [I3] can be also simplified:

[$\overline{\text{I3}}$] For each crisp input $x \in \mathcal{X}$, the output $\varphi_\Theta^{\text{CFR}}(x)$ belongs to the convex hull of consequents C_i of all rules such that $x \in \operatorname{Supp}(A_i)$ (i.e., all firing rules).

Property [I4] remains unchanged.

Preprocessing means that instead of the original rule base Θ we have to use a *modified* rule base $\overline{\Theta} = (\overline{A}_i, \overline{C}_i)_{i=1}^n$ given by (16), (18). This rule base $\overline{\Theta}$ has to be downloaded into the Mamdani–Assilian controller. Nevertheless, the input-output correspondence satisfies the desired properties [I1–3] with respect to the *original* rule base $\Theta = (A_i, C_i)_{i=1}^n$. It is usually impossible to obtain this behavior by application of a Mamdani–Assilian controller directly to the rule base Θ.

8 Conclusion

Both theoretical motivation and practical experiments suggest that the controller with conditionally firing rules [10] allows us to enhance the performance in comparison to a Mamdani–Assilian controller without a change of the rule base. Its software implementation requires just adding three easy transformations. These require additional calculations, but do not extend the order of complexity.

Our treatment applies to an arbitrary finite dimension of the input and output spaces. This approach is now ready for further tests on practical control tasks.

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