A universal integral

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Abstract

Based on a minimal set of axioms we introduce a general integral which can be defined on arbitrary measurable spaces. It acts on measures which are only (finite) monotone set functions and on measurable functions whose range is contained in the unit interval. We introduce the notion of integral equivalence of pairs of measures and functions which leads us to a special important general integrals called universal integral. Several special types of such functionals, including extremal ones, are characterized.

Key words. General integral, monotone set function, integral equivalence, universal integral, Choquet integral, semicopula.

1 Introduction

We try to contribute to a classical discussion: "What is an integral?" Based on certain minimal sets of axioms we introduce two concepts of functionals (here called general and universal integrals), which can be defined on arbitrary measurable spaces and which act on measures which are only (finite) monotone set functions and for measurable functions whose range is contained in the unit interval. Several special types of such functionals, including extremal ones, are characterized.

Clearly, the Choquet and the Sugeno integral are well-known examples of universal integrals: for any monotone set function m vanishing at the empty set and for any measurable function f with range $[0,1]$ the Choquet integral (considered earlier in [19]) is given by

$$\mathbf{Ch}(f) = \int_0^1 m(\{x \mid f(x) \geq t\})\, dt.$$

Note that the original proposal in [3] was done for special monotone set functions (capacities) only. The Sugeno integral (originally considered for continuous monotone set functions) is given by

$$\mathbf{Su}(f) = \sup_t \min(t, m(\{x \mid f(x) \geq t\})),$$

and again it can be defined for any m and f.

Usually these integrals are either interpolations (when sets are represented as characteristic functions), e.g., Choquet integral on $[0,1]$, or extrapolations, e.g., the Choquet integral on $[0,\infty]$. We will concentrate on interpolative integrals trying to provide some common framework. Based on a natural (although very weak) axiomatization we shall prove some intrinsic properties.

2 Integral equivalence

Throughout of this paper, let X be a fixed non-empty set, $\mathcal{A}$ a σ-algebra of subsets of X (in the case of a finite set X we usually take $\mathcal{A} = 2^X$), and $\mathcal{F}$ the class of all measurable functions $f\colon X \to [0,1]$. Finally, denote by $\mathcal{M}$ the class of all monotone set functions $m\colon \mathcal{A} \to [0,1]$ (considered, sometimes with additional properties, in [4,6,11,15,18,20]) which satisfy $m(\emptyset) = 0$, $m(X) = 1$ and $m(A) \leq m(B)$ whenever $A \subseteq B$.

Since we will consider integrals depending only on the pair consisting of a measure and a function we introduce a generalization of the equality almost everywhere of two functions.

Definition 2.1 Let $m_1, m_2 \in \mathcal{M}$ and $f_1, f_2 \in \mathcal{F}$. We say that the pairs (m_1, f_1) and (m_2, f_2) are *integral equivalent*, in symbols $(m_1, f_1) \sim (m_2, f_2)$, whenever

$$m_1(\{x \mid f_1(x) \geq t\}) = m_2(\{x \mid f_2(x) \geq t\})$$

for every $t \in [0,1]$.

Remark 2.2

(i) Fixing the measures $m_1 = m_2 = m$ in Definition 2.1 we obtain the m-indistinguishability from

[17] of functions f_1 and f_2, and in this case we will write $f_1 \sim_m f_2$.

(ii) Fixing the functions $f_1 = f_2 = f$ in Definition 2.1 we can introduce the f-indistinguishability of measures m_1 and m_2, and in this case we will write $m_1 \sim_f m_2$.

Example 2.3 (i) Let m be a probability measure. Then the classical equality m-almost everywhere of two functions f and g implies that $f \sim_m g$, but not vice versa, i.e., the relation $\sim_m$ is a generalization of the equality m-almost everywhere.

(ii) Note that for the weakest monotone measure $m_* \colon \mathcal{A} \to [0,1]$ given by

$$m_*(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{otherwise,} \end{cases}$$

we have $f_1 \sim_{m_*} f_2$ if and only if $\inf f_1 = \inf f_2$.

(iii) Similarly, for the strongest monotone measure $m^* \colon \mathcal{A} \to [0,1]$ given by

$$m^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

we have $f_1 \sim_{m^*} f_2$ if and only if $\sup f_1 = \sup f_2$ and both function either simultaneously possess the maximal value, or not. Hence, if X is finite, then $f_1 \sim_{m^*} f_2$ if and only if $\max f_1 = \max f_2$.

(iv) We get $m_1 \sim_{\mathbf{c}} m_2$ for all $m_1, m_2 \in \mathcal{M}$ and each constant function $\mathbf{c} \colon X \to [0,1]$ given by $\mathbf{c}(x) = c$. On the other hand, if $f = c \cdot \mathbf{1}_A$ for some $c \in \,]0,1]$ and $A \in \mathcal{A}$, then we have $m_1 \sim_f m_2$ if and only if $m_1(A) = m_2(A)$.

Proposition 2.4 *Define for $m \in \mathcal{M}$ and for a measurable function $\varphi \colon X \to X$ the measure $m^\varphi \in \mathcal{M}$ by $m^\varphi(A) = m(\varphi^{-1}(A))$. Then for all $f \in \mathcal{F}$ the pairs $(m, f \circ \varphi)$ and (m^φ, f) are integral equivalent.*

We shall compare the notion of m-indistinguishability with the equality almost everywhere based on the notion of null set for the case of monotone set functions given in [9], see [11].

Definition 2.5 A set $N \in \mathcal{A}$ is a *null set* with respect to $m \in \mathcal{M}$ if for all $A \in \mathcal{A}$ we have

$$m(A \cup N) = m(A).$$

Definition 2.6 Let $m \in \mathcal{M}$ and $f, g \in \mathcal{F}$. We say that $f = g$ *almost everywhere with respect to* m if there exists a null set $N \in \mathcal{A}$ with respect to m such that $f(x) = g(x)$ for every $x \in \complement N$.

Theorem 2.7 *Let $m \in \mathcal{M}$ and $f, g \in \mathcal{F}$. If $f = g$ almost everywhere with respect to m, then f and g are m-indistinguishable.*

The converse of Theorem 2.7 is not true in general:

Example 2.8 The only null set with respect to m^* is the empty set. Therefore we have $f_1 = f_2$ almost everywhere with respect to m^* if and only if $f_1(x) = f_2(x)$ for every $x \in X$, but $f_1 \sim_{m^*} f_2$ if and only if $\sup f_1 = \sup f_2$.

3 General integral

We have introduced in the paper [8] the notion of general integral.

Definition 3.1 A mapping $\mathbf{G} \colon \mathcal{M} \times \mathcal{F} \to [0,1]$ is called a *general integral* on $(X, \mathcal{A})$ if it satisfies the following conditions:

(I1) *boundary conditions*, i.e., for each $m \in \mathcal{M}$ we have

$$\mathbf{G}(m, 0) = 0 \quad \text{and} \quad \mathbf{G}(m, 1) = 1,$$

(I2) *monotonicity* in both coordinates, i.e., for $m_1 \leq m_2$ and $f_1 \leq f_2$ we have

$$\mathbf{G}(m_1, f_1) \leq \mathbf{G}(m_2, f_2),$$

(I3) *extension of the measure*, i.e., for each $A \in \mathcal{A}$ and for each $m \in \mathcal{M}$ we have

$$\mathbf{G}(m, \mathbf{1}_A) = m(A).$$

(I4) *idempotency*, i.e., for every $c \in [0,1]$ we have

$$\mathbf{G}(m, c) = c,$$

(I5) *existence of a pseudo-multiplication*, i.e., there exists a binary operation $\otimes \colon [0,1]^2 \to [0,1]$ such that for each $m \in \mathcal{M}$, each $c \in [0,1]$ and each $A \in \mathcal{A}$

$$\mathbf{G}(m, c \cdot \mathbf{1}_A) = c \otimes m(A),$$

(I6) for each measurable function $\varphi \colon X \to X$, and for each $(m, f) \in \mathcal{M} \times \mathcal{F}$ we have

$$\mathbf{G}(m^\varphi, f) = \mathbf{G}(m, f \circ \varphi),$$

where the measure $m^\varphi \in \mathcal{M}$ is given by $m^\varphi(A) = m(\varphi^{-1}(A))$.

Obviously, (I3) as well as (I4) implies (I1); however we prefer to keep axiom (I1) in order to stress the boundary conditions of integral functionals.

Observe that one of the consequences of axiom (I6) is that for a general integral $\mathbf{G}$ and a Dirac measure

$m_{\{x_0\}}$ (all the mass is concentrated in $x_0 \in X$) we have $\mathbf{G}(m_{\{x_0\}}, f) = f(x_0)$ for each $f \in \mathcal{F}$.

Also, the class $\mathcal{G}$ of general integrals is convex, and it is closed under each idempotent aggregation operator.

Observe that the axioms (I1)–(I4) imply that the pseudo-multiplication $\otimes$ required in axiom (I5) is a semicopula (see [1, 5]).

Following the ideas of inner and outer measures in classical measure theory, we obtain the following result:

Theorem 3.2 *Let $\otimes$ be a semicopula. Then the class $\mathcal{G}_\otimes$ of all general integrals related to $\otimes$ is a convex class with smallest element $\mathbf{G}_\otimes$ and greatest element $\mathbf{G}^\otimes$, given by*

$$\begin{aligned}\mathbf{G}_\otimes(m, f) &= \sup\{t \otimes m(\{f \geq t\}) \mid t \in [0,1]\},\\ \mathbf{G}^\otimes(m, f) &= (\sup f) \otimes m(\{f > 0\}).\end{aligned}$$

Recall that the drastic product $T_\mathbf{D}$ is the weakest and the minimum $T_\mathbf{M}$ is the strongest semicopula. Obviously, for any two semicopulas $\otimes_1$ and $\otimes_2$ with $\otimes_1 \leq \otimes_2$ we have $\mathbf{G}_{\otimes_1} \leq \mathbf{G}_{\otimes_2}$ and $\mathbf{G}^{\otimes_1} \leq \mathbf{G}^{\otimes_2}$.

Corollary 3.3 *The smallest general integral $\mathbf{G}_* = \mathbf{G}_{T_\mathbf{D}}$ and the largest general integral $\mathbf{G}^* = \mathbf{G}_{T_\mathbf{M}}$ are given by*

$$\begin{aligned}\mathbf{G}_*(m, f) &= \sup\{T_\mathbf{D}(t, m(\{f \geq t\})) \mid t \in [0,1]\}\\ &= \max(\mathrm{essinf}_m f, m(\{f = 1\})),\\ \mathbf{G}^*(m, f) &= \min(\sup f, m(\{f > 0\})).\end{aligned}$$

Here $\mathrm{essinf}_m f = \sup\{t \in [0,1] \mid m\{f \geq t\} = 1\}$.

Note that the Choquet and the Sugeno integral are examples of general integrals. Moreover, these integrals (and also $\mathbf{G}_*$) give the same result for integral equivalent pairs (m_1, f_1) and (m_2, f_2).

However, a general integral $\mathbf{G}$ does not fulfill $\mathbf{G}(m_1, f_1) = \mathbf{G}(m_2, f_2)$ whenever $(m_1, f_1) \sim (m_2, f_2)$, in general.

Example 3.4 Let $X =]0, 1[$ and $\mathcal{A}$ the σ-algebra of Borel subsets of X. Define $\mathbf{G} : \mathcal{M} \times \mathcal{F} \to [0,1]$ in the following way

$$\mathbf{G}(m, f) = m(f > 0) \cdot \sup f.$$

Observe that $\mathbf{G} = \mathbf{G}^{T_\mathbf{P}}$ is the strongest general integral based on the product $T_\mathbf{P}$ as the semicopula $\otimes$. Then $(m_*, \mathbf{0}) \sim (m_*, \mathrm{id}_X)$, but $\mathbf{G}(m_*, \mathbf{0}) = 0$ and $\mathbf{G}(m_*, \mathrm{id}_X) = 1$.

4 A universal integral

To overcome the possibility of obtaining different outputs applying a general integral to integral equivalent pairs $(m_1, f_1) \sim (m_2, f_2)$, we introduce universal integrals.

Definition 4.1 A general integral $\mathbf{U}\colon \mathcal{M} \times \mathcal{F} \to [0,1]$ is called *universal integral* whenever for each integral equivalent pairs (m_1, f_1) and (m_2, f_2) from $\mathcal{M} \times \mathcal{F}$ we have

$$\mathbf{U}(m_1, f_1) = \mathbf{U}(m_2, f_2).$$

Note that there are two other equivalent concepts of defining universal integrals [7, 17].

Again, the Choquet, and the Sugeno integral are examples of universal integrals. The class $\mathcal{U}$ of universal integrals is also a convex set, and it is closed under any idempotent aggregation operator.

A generalization of the construction method of Choquet and Sugeno integrals for a universal integral can be described in the following way (see [17]). Suppose that for a general integral $\mathbf{U}\colon \mathcal{M} \times \mathcal{F} \to [0,1]$ there exists a monotone function $\mathbf{J}\colon \mathcal{L}([0,1]) \to [0,1]$ such that

$$\mathbf{U}(m, f) = \mathbf{J}(h_{m,f}), \tag{1}$$

where $\mathcal{L}([0,1])$ is the class of all Borel measurable functions from $[0,1]$ to $[0,1]$ and $h_{m,f}\colon [0,1] \to [0,1]$ is given by $h_{m,f}(t) = m(\{f \geq t\})$. Then evidently $\mathbf{U}$ is a universal integral (observe that $(m_1, f_1) \sim (m_2, f_2)$ if and only if $h_{m_1,f_1} = h_{m_2,f_2}$). Conversely, for a universal integral $\mathbf{U}$ it is enough to put

$$\mathbf{J}(g) = \sup\{\mathbf{U}(m, f) \mid h_{m,f} \leq g\}$$

in order to obtain (1).

Now we recall two properties of the class $\mathcal{U}$ of universal integrals, see also [17]:

(i) For each $\mathbf{U} \in \mathcal{U}$ we have

$$\mathbf{U}_* \leq \mathbf{U} \leq \mathbf{U}^*,$$

where $\mathbf{U}_*$ and $\mathbf{U}^*$ are given by

$$\begin{aligned}\mathbf{U}_* &= \mathbf{G}_*,\\ \mathbf{U}^*(m, f) &= \min(\mathrm{essup}_m f, m(\{f > 0\})),\end{aligned}$$

and $\mathrm{essup}_m f = \sup\{t \in [0,1] \mid m\{f \geq t\} > 0\}$.

(ii) For each measurable space $(X, 2^X)$, each $\{0,1\}$-valued measure $m \in \mathcal{M}$ and each $f \in \mathcal{F}$ we have $\mathbf{U}_*(m, f) = \mathbf{U}(m, f) = \mathbf{U}^*(m, f)$ for all $\mathbf{U} \in \mathcal{U}$. Defining the function $L_m\colon \mathcal{F} \to [0,1]$ by $L_m(f) = \mathbf{U}_*(m, f)$, then L_m is a lattice polynomial on X, and it can be written as

$$L_m(f) = \bigvee_{m(A)=1} \bigwedge_{x \in A} f(x).$$

For universal integrals we have the following counterpart of Theorem 3.2:

Theorem 4.2 *Let $\otimes$ be a semicopula. Then the class $\mathcal{U}_\otimes$ of all universal integrals related to $\otimes$ is a convex class with smallest element $\mathbf{U}_\otimes$ and greatest element $\mathbf{U}^\otimes$, given by*

$$\begin{aligned}\mathbf{U}_\otimes &= \mathbf{G}_\otimes,\\ \mathbf{U}^\otimes(m,f) &= (\mathrm{essup}_m f)\otimes m(\{f>0\}).\end{aligned}$$

Observe that $\mathbf{U}_{T_\mathbf{M}}$ is the Sugeno integral, while $\mathbf{U}_{T_\mathbf{P}}$ is the Shilkret integral [13] (originally defined for maxitive measures only).

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