# Dealing with Incompleteness of Preferences in Group Decision Making Problems

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## Abstract

While considering processes of decision making we often encounter the problem of incomplete information. In group decision making (GDM) problems each decision maker is supposed to provide a matrix that describes his/her preferences over the set of given options. However, we have to take into account that an expert can not be able to define his/her preferences about all the options. Usually the problem is solved by an additional phase of estimating missing values. In the article we want to suggest and discuss a totally different approach that consists in adopting GDM algorithm so as it can deal with incomplete preference matrices.

**Keywords:** group decision making, incomplete preference relations, Atanassov's intuitionistic fuzzy sets.

## 1 Introduction

Group decision making (GDM) procedures are used in processes of choosing one best option from the set of options. A central notion of these procedures is a preference matrix that models preferences of a decision maker. Possible options are compared pairwise, and to each pair a value, that represent a degree to which one option is preferred over the other, is assigned. It is usually assumed that such a value is given for every pair of options, forming a complete preference relation. However, worth considering is a case, when some of these values are missing for some reason like the lack of sufficient knowledge of an expert, impossibility of comparing some options under particular criterion, ignoring an option by an expert or simply loss of some information during the process. Algorithms of GDM should thus be able to deal with incomplete knowledge.

Different approaches to that problem have been proposed in the literature, most of which try to complete somehow the incomplete preference matrices and then to apply a GDM algorithm. However, it is not always possible (and probably not always necessary) to fill all the missing values. In this article we like to suggest a different approach, namely: we leave the missing values not filled, and adopt the GDM algorithms to deal with incomplete preference relations. We examine and compare these two approaches, and discuss advantages and drawbacks of both of them.

## 2 Main aspects of GDM problems

First let us briefly describe the main idea of the class of GDM algorithms based on IF-sets (Atanassov's intuitionistic fuzzy sets, see [1]) with triangular norms. In the following, we focus on preference relations used in those algorithms.

The problem of group decision making consists in finding one compromise option that best suits the preferences of the whole group of decision makers. By $\boldsymbol{P} = \{p_1, ..., p_m\}$ we denote a set of $m \geq 1$ *decision makers (experts)*, and $\boldsymbol{S} = \{s_1, ..., s_n\}$ is a set of $n \geq 2$ *options*. Each expert $p_k$ constructs his/her preference matrix $R_k = [r_{ij}^k]$ with $i, j = 1, 2, ..., n$, and $r_{ij}^k \in [0, 1]$, which represents his/her preferences about options from $\boldsymbol{S}$. Finally, $Q$ denotes a relative linguistic quantifier of "most"-type (see [6]).

The process of GDM can be performed in two ways, as proposed by Kacprzyk in [6]: directly and indirectly.

In the direct approach a solution $S_Q$ is obtained directly from preference matrices:

$$\{R_1, ..., R_m\} \to S_Q.$$

$S_Q$ is then a fuzzy set of options such that a soft majority, $Q$ experts, are not against them.

When applying the indirect approach the individual matrices $R_1, ..., R_m$ are first aggregated into one group preference matrix:

$$\{R_1, ..., R_m\} \to R \to W_Q$$

and next the solution $W_Q$ is constructed. It is a fuzzy set of options that are preferred to $Q$ other options. This solution is known also as a fuzzy $Q$-consensus winner ([6]).

Depending on the used t-norm we divide the algorithms into another two subclasses. The first one refers to $\boldsymbol{t}$ being a nonstrict Archimedean t-norm with a negation induced by $\boldsymbol{t}$ $(\nu = \nu_{\boldsymbol{t}})$, and the second one - to $\boldsymbol{t}$ being a strict t-norm or $\boldsymbol{t} = \wedge$ with the Łukasiewicz negation $\nu = \nu_L$. This distinction is important from the point of view of GDM as it results in different semantic interpretation of a hesitation index. When using non-strict Archimedean t-norms hesitation is interpreted in classical way, but when using strict t-norm or $\boldsymbol{t} = \wedge$ hesitation index combines hesitation with fuzziness. For more details see [9]. In the article we restrict ourselves to the case of a nonstrict Archimedean t-norm and indirect approach.

### 2.1 Modeling preferences on the basis of IF-set theory with triangular norms

As already mentioned, each decision maker provides his/her preferences by means of a fuzzy preference relation $R_k : \boldsymbol{S} \times \boldsymbol{S} \to [0, 1]$ that can be represented by a $n \times n$ preference matrix $R_k$. Each element $r_{ij}^k$ of that martix is a *degree to which an expert $p_k$ prefers an option $s_i$ over an option $s_j$*. Of course, $r_{ij}^k \in [0, 1]$; $r_{ij}^k = 0.5$ indicates indifference between $s_i$ and $s_j$. Clearly, the greater the value of $r_{ij}^k$, the higher the preference of $s_i$ over $s_j$.

It is usually assumed that $r_{ij}^k = 1 - r_{ji}^k$. However, this restriction enables to model only imprecision (vagueness) of preferences, but not uncertainty (hesitation of an expert). The theory that gives a way to incorporate uncertainty is the Atanassov's intuitionistic fuzzy set theory (IF-set theory). An IF-set $\mathcal{E}$ is a pair of fuzzy sets: $\mathcal{E} = (A, A^d)$, where $A$ is as fuzzy set of elements that belong to $\mathcal{E}$, and $A^d$ is dual to $A$ and contains elements that do not belong to $\mathcal{E}$. In the classical fuzzy set theory ([13] ) $A^d = A'$, where $A'(x) = 1 - A(x)$ for each $x$. In IF-set theory $A^d \subset A'$, so a margin of uncertainty is left. It is called *hesitation index*, and is defined as:

$$\chi_{\mathcal{E}}(x) = 1 - A(x) - A^d(x) \qquad (1)$$

for each $x$.

As presented in [8, 9] the theory of IF-sets can be generalized by using arbitrary strong negation $\nu$ instead of the Łukasiewicz negation and arbitrary t-norm $\boldsymbol{t}$ instead of the Łukasiewicz t-norm. Thus we rewrite the main condition as $A^d \subset A^\nu$. The formula (1) will take the form:

$$\chi_{\mathcal{E}}(x) = \nu(A(x)\, \boldsymbol{t}^{\,\nu} A^d(x)) = \nu(A(x))\, \boldsymbol{t}\, \nu(A^d(x)), \quad (2)$$

where $\boldsymbol{t}^\nu$ is a t-conorm defined as $a\boldsymbol{t}^\nu b = \nu(\nu(a)\boldsymbol{t}\nu(b))$.

Applying this theory to the preference modeling, we will replace the condition $r_{ij}^k = 1 - r_{ji}^k$ with more flexible one $r_{ij}^k \leq 1 - r_{ji}^k$, or, in general, to $r_{ij}^k \leq \nu(r_{ji}^k)$, what gives us the possibility to model hesitation of the decision maker in a flexible way. That hesitation is calculated for each expert as

$$h_{ij}^k = \nu(r_{ij}^k)\, \boldsymbol{t}\, \nu(r_{ji}^k) \qquad (3)$$

and stored in a hesitation matrix $H_k = [h_{ij}^k]$.

### 2.2 The algorithm of GDM

Here we present the algorithm of GDM for nonstrict Archimedean t-norms and indirect approach. Functions $f, f^*, g, g^*$ are cardinality patterns (see [12]).

**Step 1** Aggregate individual preference matrices into one group preference matrix $R = [r_{ij}]$:

$$r_{ij} = \frac{1}{m} \sum_{k=1}^{m} f(r_{ij}^k).$$

A value $r_{ij}$ represents a degree to which an option $s_i$ is preferred over an option $s_j$ by the whole group of decision makers.

**Step 2** Calculate a degree $r_i$ to which an option $s_i$ is preferred over all the other options:

$$r_i = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} f^*(r_{ij})$$

**Step 3** Construct hesitation matrices $H_k = [h_{ij}^k]$ according to (3). Next, aggregate them into one group hesitation matrix $H = [h_{ij}]$:

$$h_{ij} = \frac{1}{m} \sum_{k=1}^{m} g(h_{ij}^k).$$

**Step 4** Calculate a degree $h_i$ to which a whole group is hesitating whether to choose an option $s_i$ or some other option:

$$h_i = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} g^*(h_{ij}).$$

**Step 5** Compute $Q(r_i)$ and $Q(r_i \boldsymbol{t}^\circ h_i)$ for $i = 1, 2, ..., n$ that is a degree to which an option $s_i$ is preferred over

$Q$ other options by the group of experts, taking into account also hesitation of the group.

**Step 6** The final solution is:

$$W_Q = [Q(r_1), Q(r_1 \boldsymbol{t}^{\circ} h_1)]/s_1 + ... \\ ... + [Q(r_n), Q(r_n \boldsymbol{t}^{\circ} h_n)]/s_n.$$

## 3 Incompleteness in GDM problems

So far we required an expert to compare every pair of options. This requirement however is not always possible to fulfil. The reasons could be varied: an expert may consider some options uninteresting for him/her for some reason, or may have only partial knowledge about some certain aspects or about a subset of options. It is important to distinguish between this situation and the situation when two options were considered indifferent (i.e. $r^k_{ij} = r^k_{ji} = 0.5$). The case of a missing value rather resembles the situation when $r^k_{ij} = 0$ and $r^k_{ji} = 0$. However, an expert should have a possibility to express his/her lack of knowledge, and must not be forced to give every value of $R_k$ if he/she is not able to do this - such a demand could lower information quality.

The concept that allows to express the situation, when some values $r^k_{ij}$ are missing is the concept of *incomplete fuzzy preference relation* $R_k$. It is defined as a fuzzy set $R_k : \boldsymbol{S} \times \boldsymbol{S} \to [0, 1]$ characterized by a partial membership function. By *a partial function* we mean a function which is not defined for some of its domain.

An undefined value of a fuzzy preference relation $R_k$ will be denoted by a symbol *, and will be called a *missing value.* A set

$$MR_k = \left\{(s_i, s_j) \in R_k : r^k_{ij} = *\right\} \quad (4)$$

is a set of all pairs of options for which preference values were not defined by an expert $p_k$.

From now on, to avoid any misunderstanding, the fuzzy preference relation with no missing values will be called *complete* fuzzy preference relation.

As GDM algorithms were constructed for complete fuzzy preference matrices, we have to introduce some new methods to deal with incomplete fuzzy preference matrices. There are two main possible approaches we can apply:

- complete missing values:
  - with some neutral value, like 0.5, or a mean value;
  - according to some external information, e.g. taken from preference relation of another expert(s), or by interacting with an expert;
  - according to some additional conditions - usually transitivity condition;
- do not complete missing values and adopt the algorithm of GDM so as it can operate on incomplete preference matrices.

The first approach is wildly discussed in the literature (see e.g. [2, 3, 4, 5]). We would like to point out some problems connected with it:

- Estimating missing values according to some additional property, e.g. transitivity property, is possible only under certain conditions. Moreover, as experimental studies show (see [11]), in real-life situation preference relations are inconsistent and violate transitivity condition. In such cases estimated values may not reflect the real intention of the expert.
- Interactive procedures that cooperate with an expert during the process of completing missing values, are often very hard to execute.
- When using some external knowledge, e.g. from another experts, we do not take into account the specific character of a particular expert.
- Filling missing values with a mean, random or neutral value is very simple but too rigid.
- All of these techniques introduce some additional information that was not provided in the original preference matrix, thus can disturb its meaning.

Completing missing values is particularly difficult if a degree of incompleteness of a preference matrix is high. In such a case it is very hard to complete all missing values in a reasonable way.

Therefore, we like to propose a second approach that do not change preference matrices but an algorithm itself. We belief that such an approach can form a good alternative for estimating in some cases of GDM problems. The advantages of this approach are, among others:

- We do not have to process any additional calculation to estimate missing values.
- We do not disrupt the original preference relations, and we are not loosing the information which *lack of information* also is (as we have mentioned, inserting some values in place of empty values can distort the intention of an expert, as we add some extra information that an expert didn't actually provide).

- We are able to complete the process of solving GDM problem even if we have no possibility to estimate all of the missing values.

Such an approach generates of course some new problems, such as:

- The solution of the algorithm could be in some cases incomplete - it would happen if we had not enough information about some option.
- We are not using any additional knowledge that could have been helpful.

The group of algorithms of GDM for incomplete fuzzy preference relations was developed and one of the algorithm, for nonstrict Archimedean t-norms and for indirect approach, is presented in the next subsection.

### 3.1 An algorithm of GDM for incomplete preference matrices

As we defined in (4), $MR_k$ is a set of pairs of options for which an expert $p_k$ did not define his/her preference. At the beginning all values $r_{ij}^k$ that are missing, but for which $r_{ji}^k$ is given and $r_{ji}^k \geq a^*$, are calculated as $\nu(r_{ji}^k)$ and are no longer considered as missing. $a^*$ is a fixed point of a negation $\nu$ (for Łukasiewicz negation $a^* = 0.5$).

Below we present GDM algorithm for nonstrict Archimedean t-norms and indirect approach.

**Step 1**

$$E_{ij} = \left\{(s_i, s_j) \in R_k : r_{ij}^k = *, k = 1, 2, ..m\right\},$$

$$r_{ij} = \begin{cases} \frac{1}{m-|E_{ij}|} \sum_{k=1}^{m} f(r_{ij}^k), & \text{if } e_{ij} < m \\ * & \text{otherwise.} \end{cases}$$

**Step 2**

$$E_i = \{(s_i, s_j) \in R : r_{ij} = *, j = 1, 2, ...n\},$$

$$r_i = \begin{cases} \frac{1}{n-1-|E_i|} \sum_{\substack{j=1 \\ j \neq i}}^{n} f^*(r_{ij}), & \text{if } e_i < n-1 \\ * & \text{otherwise.} \end{cases}$$

**Step 3**

$$h_{ij} = \begin{cases} \frac{1}{m-|E_{ij}|} \sum_{k=1}^{m} g(h_{ij}^k), & \text{if } e_{ij} < m \\ * & \text{otherwise.} \end{cases}$$

**Step 4**

$$h_i = \begin{cases} \frac{1}{n-1-|E_i|} \sum_{\substack{j=1 \\ j \neq i}}^{n} g^*(h_{ij}), & \text{if } e_i < n-1 \\ * & \text{otherwise.} \end{cases}$$

**Step 5** For all defined $r_i$ and $h_i$ compute $Q(r_i)$ and $Q(r_i \boldsymbol{t}^\circ h_i)$ for $i = 1, 2, ..., n$. Put $Q(r_i) = *$ and $Q(r_i \boldsymbol{t}^\circ h_i) = *$ otherwise.

**Step 6** Finally,

$$W_Q = [Q(r_1), Q(r_1 \boldsymbol{t}^\circ h_1)]/s_1 + ... \\ ... + [Q(r_n), Q(r_n \boldsymbol{t}^\circ h_n)]/s_n.$$

It is easy to notice that the solution may be incomplete. It is also important to be aware of the fact that the solution could possibly be obtained from incomplete information. It seems useful to introduce a factor that will signify a degree of completeness for a given option. It will be determined as a ratio of number of missing values involving option $s_i$ to all preference values involving $s_i$. So, we define:

$$MR_k^i = \{(s_i, s_j) \in MR_k \cup (s_j, s_i) \in MR_k, j = 1, 2, ..., n)\}$$

and the *completness factor* of an option $s_i$ is

$$CF_i = 1 - \frac{\sum_{k=1}^{m} MR_k^i}{m(2n-2)}. \quad (5)$$

Obviously, $CF_i \in [0, 1]$. $CF_i = 0$ means total incompleteness (no preferences were defined for any pair of options involving $s_i$), and $CF_i = 1$ means total completeness (i.e. option $s_i$ was compared with all the other options).

## 4 An example of solving GDM problem with incomplete preference relations

A company X decided to introduce a new report software. Experts from three departments, $p_1, p_2$ and $p_3$, were asked to compare and assess five different reporting tools, $s_1, s_2, s_3, s_4$ and $s_5$, in order to choose and apply the best one.

As a result, the experts provided three incomplete matrices describing their preferences about given options of the software:

$$R_1 = \begin{bmatrix} 0.5 & * & 0 & * & * \\ * & 0.5 & * & * & * \\ 0.9 & * & 0.5 & * & 0.2 \\ * & * & * & 0.5 & 0.3 \\ 1 & * & 0.7 & 0.7 & 0.5 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.5 & * & * & * & * \\ * & 0.5 & * & * & * \\ * & * & 0.5 & * & 0.5 \\ 0.9 & * & * & 0.5 & 0.6 \\ * & * & 0.5 & 0.4 & 0.5 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.5 & * & * & * & * \\ * & 0.5 & * & * & * \\ * & * & 0.5 & * & * \\ * & * & * & 0.5 & * \\ 1 & * & * & 1 & 0.5 \end{bmatrix}.$$

Each expert defined the preferences according to his/her knowledge, personal experience, and taking into account the specific needs and tasks of their departments. That is why their preferences differ to a large extend from each other and are not complete.

We have to emphasize that incompleteness of the preference matrices cannot be treated here as an aberration or a defect that has to be fixed, but is a natural property resulting from the fact that every expert is a specialist only in some part of the problem and has a very subjective point of view. Therefore, completing missing values in this case seems to be unjustified and improper. As matrices are incomplete to high degree it is also difficult or even impossible to complete them in a reasonable way.

Instead of completing missing values, we apply the version of GDM algorithm for incomplete preference matrices presented in the previous section. For simplicity, we assume that that $f = f^* = g = g^* = id$, $\boldsymbol{t} = \boldsymbol{t}_L$, $\nu = \nu_L$, $Q = id$ (setting these parameters to different values is reasonable when we try to optimize the algorithm for a specific application).

First, we make use of a property of a preference matrix saying that $r^k_{ij} \leq \nu(r^k_{ji})$, and for all missing values $r^k_{ij}$ for which $r^k_{ji} \geq a^*$ ($a^* = 0.5$) we calculate $r^k_{ij}$ as $1 - r^k_{ji}$.

Next, individual preference matrices are aggregated into one group preference matrix that will combine the knowledge of the experts. This matrix can be complete or incomplete, but its degree of completeness will be the same or higher then a degree of completeness of individual matrices. This feature is a result of an algorithm that puts together information from all the experts and completes missing values in one preference matrix with values from another preference matrix, if given.

The group preference matrix is presented below:

$$R = \begin{bmatrix} 0.5 & * & 0 & 0.1 & 0 \\ * & 0.5 & * & * & * \\ 0.9 & * & 0.5 & * & 0.35 \\ 0.9 & * & * & 0.5 & 0.3 \\ 1 & * & 0.6 & 0.7 & 0.5 \end{bmatrix}.$$

At the same time, we calculate individual hesitation matrices and a group hesitation matrix, which is presented below:

$$H = \begin{bmatrix} 0.5 & * & 0.1 & 0 & 0 \\ * & 0.5 & * & * & * \\ 0.1 & * & 0.5 & * & 0.05 \\ 0 & * & * & 0.5 & 0 \\ 0 & * & 0.05 & 0 & 0.5 \end{bmatrix}.$$

The last aggregation step for each option returns a degree to which this option is preferred over all other options according to the whole group of experts. Finally, we get the solution $W_Q$:

$$W_Q = (0.03, 0.07)/s_1 + (*)/s_2 + (0.63, 0.70)/s_3 \\ +(0.60, 0.60)/s_4 + (0.77, 0.78)/s_5.$$

The option $s_5$ was thus chosen by experts to be the best one.

The solution is incomplete (there is no value assigned to option $s_2$) as none of the experts said anything about the reporting tool $s_2$. That might suggest that they do not know this tool or consider it highly improper and uninteresting for the company. It may be treated as a hint that this option should no longer be taken into consideration.

To get more information about the quality of the solution, we calculate additional factor for each option, $CF_i$, according to (5) (in the calculation we excluded option $s_2$):

$$CF_1 = 0.44, \\ CF_3 = 0.33, \\ CF_4 = 0.44, \\ CF_5 = 0.77.$$

The option $s_5$ was compared by experts with most of other options, thus a value of $CF_5$ is quite high. Therefore, we may conclude that the choice of this option as the best one is right and well motivated.

## 5 Conclusions

The article presents a new method for dealing with incompleteness of information in GDM problems. This approach, unlike those that are usually discussed in the literature, does not fill missing values. Instead, it changes the very GDM algorithm. We pointed out some difficulties connected with estimating missing values and we shown an application of new version of algorithm for incomplete preference relations. We considered the case when the degree of incompleteness is too high to estimate or complete missing values in a reasonable way. In such cases the proposed modified algorithm is a worth considering alternative tool.

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