On a Proximity-Based Tolerant Inclusion[1]

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Abstract

This paper is devoted to an extension of the inclusion operator. The main idea that we suggest is to relax the arguments of the inclusion by means of proximity-based modifiers in order to obtain a more tolerant inclusion. The use of such a relaxed inclusion is illustrated in the framework of databases by the extension of the division operator.

Keywords: inclusion, tolerant inclusion, proximity relation, relational division.

1 Introduction

Several types of extensions of the inclusion have been proposed in the "fuzzy set research community" in order to either: i) define the inclusion when fuzzy sets come into play, or ii) to make the result of the inclusion more flexible, i.e., valued in [0, 1], or iii) to authorize different types of exceptions. In the first case, Zadeh [11] defined the inclusion of fuzzy sets in the following way: $E \subseteq F \Leftrightarrow \forall x \in U, E(x) \leq F(x)$, where U denotes the universe of discourse. In the second case, the objective is to discriminate between situations significantly different where the usual inclusion does not hold, and a solution consists in using a fuzzy implication to define the graded inclusion [1]. In the third case, two visions of exceptions have been considered so far, which lead to two types of approximate inclusion indicators. A first idea, developed in [3], consists in weakening the universal quantifier underlying the inclusion into "almost all", in the perspective of defining an *approximate* inclusion. The basic idea is to tolerate, in the evaluation of $E \subseteq F$, a certain number of exceptions (i.e., of elements of E which are not totally included in F according to a given implication), and in that sense the corresponding approximate inclusion can be called a *quantitative* one. In [4], another way of defining an approximate inclusion is presented and the idea is rather to give a central role to the intensity of the exceptions in order to define an inclusion indicator that can ignore to a certain extent the "low intensity" ones. In that sense, the operator defined can be called a *qualitative* approximate inclusion.

Here, the idea is to take into account the notion of closeness between the elements of the domain, so as to define a *proximity-based tolerant inclusion*. For instance, one may consider that a set E is included in a set F if, for every element x of E, x is present in F (classical inclusion) or if F contains an element *close* to x.

The rest of the paper is structured as follows. In section 2, we recall some basic notions related to the inclusion of fuzzy sets in the sense of Zadeh [11], as well as the way of defining proximity-based modifiers. In section 3, the principle of the proximity-based inclusion is introduced and we show how it can be applied to crisp sets as well as fuzzy sets. Section 4 illustrates the practical utility of such a tolerant inclusion in the context of database querying where a relaxation of the division operator is studied. Section 5 provides a brief comparison between the proximity-based approach and the two approximate inclusion indicators introduced in

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[3] and [4]. The last section concludes the paper. The objective and main contributions are recalled and some perspectives as to future works are discussed.

2 Preliminaries and Background

2.1 Boolean Inclusion of Fuzzy Sets

If A and B denote two crisp sets built on U, the usual way for defining the inclusion of A in B is:

$$(A \subseteq B) \Leftrightarrow (\forall x \in U, x \in A \Rightarrow x \in B) \quad (1)$$

This definition can be extended in a canonical way to two fuzzy sets A and B, which leads to:

$$(A \subseteq B) \Leftrightarrow (\forall x \in U, A(x) \rightarrow_{R\text{-}G} B(x)) \quad (2)$$

where $\rightarrow_{R\text{-}G}$ stands for Rescher-Gaines implication:

$p \rightarrow_{R\text{-}G} q = 1$ if $p \leq q$, 0 otherwise.

This view does not take into account the proximity between the elements of the universe, therefore it can happen that the result of the inclusion is false while it would be true if that notion was used.

Example 1. Let A and B be the two fuzzy sets :

$A = \{1/a + 0.6/b\}$, $B = \{1/a + 0.4/b + 0.9/c\}$.

According to formula (2), A is not included in B. However, if one has available the knowledge that the element b is very close (according to a proximity relation defined on the universe considered) to the element c (which strongly belongs to B), it may make sense to upgrade the membership degree of b to B (which corresponds to modifying B into a fuzzy set B') and then A may be included in B'. In that sense, $A \subseteq B'$ can be viewed as a relaxation of $A \subseteq B$. This is the basic idea that is developed in the following.♦

2.2 About Proximity-Based Modifiers

Let us first recall the formal definition of the concept of a *proximity relation* [9].

Definition 1. A proximity relation is a fuzzy relation R on a scalar domain U, such that for $u, v \in U$,

$R(u, u) = 1$ (*reflexivity*),

$R(u, v) = R(v, u)$ (*symmetry*).

The quantity $R(u, v)$ can be viewed as a grade of *approximate equality* of u with v.

An *absolute proximity* relation is an approximate equality relation E which can be modeled by a fuzzy relation of the form [9]:

$$E: U \times U \rightarrow [0, 1]$$
$$(u, v) \rightarrow E(u, v) = Z(u - v),$$

which only depends on the value of the difference $u - v$, and where Z, called a *tolerance indicator*, is a fuzzy interval centered in 0, such that:

i. $Z(r) = Z(-r)$, i.e., $Z = -Z$;
ii. $Z(0) = 1$;
iii. The support of Z is bounded and is of the form $[-\Omega, \Omega]$ where Ω is a positive real number.

In terms of trapezoidal membership function (t.m.f.), the parameter Z can be expressed by $(-z, z, \delta, \delta)$ with $\Omega = z + \delta$ and $[-z, z]$ represents the core of Z.

Proposition 1. Let Z_1 and Z_2 be two fuzzy intervals centered in 0 on scalar domain U. The following entailment holds:

$$Z_1 \subseteq Z_2 \Rightarrow E[Z_1] \subseteq E[Z_2].$$

The proof is straightforward. In the following, we will write $E[Z]$ to denote the absolute proximity relation E parameterized by Z. See [9] for other interesting properties of $E[Z]$.

Consider a fuzzy set F on the scalar domain U and an absolute proximity relation $E(Z)$, where Z is a tolerance indicator. The set F can be associated with a nested pair of fuzzy sets when using $E(Z)$ as a *tolerance relation*. Indeed,

i. we can build a fuzzy set $E^Z(F)$ close to F, such that $F \subseteq E^Z(F)$. This is the *dilation operation*. $E^Z(F)$ gathers the elements of F and those outside of F which are somewhat close to an element in F in the sense of E[Z].
ii. we can build a fuzzy set $E_Z(F)$ close to F, such that $E_Z(F) \subseteq F$. This is the *erosion operation*. $E_Z(F)$ gathers the elements of F such that all of their "neighbors" – i.e. those which are somewhat close to them – are in F.

These fuzzy sets can be constructed in the following way.

Dilation operation. Dilating the fuzzy set F by Z will provide a fuzzy set $E^Z(F)$ defined by:

$$E^Z(F)(s) = sup_{r\in U} \top(F(r), E[Z](s, r)) \quad (3)$$
$$= sup_{r\in U} \top(F(r), Z(s - r))$$

where $\top$ is a triangular norm.

Then, $F \subseteq E^Z(F)$ *and* $E^Z(F)$ *can be viewed as a weakened variant of* F.

Erosion Operation. Considering the meaning of $E_Z(F)$ given above, it seems natural to adopt the following definition:

$$E_Z(F)(s) = inf_{r\in U} (E[Z](s, r) \rightarrow F(r)) \quad (4)$$

where $\rightarrow$ *denotes a fuzzy implication. We will see later (in section 3.3) that the preservation of an important axiom of the inclusion constrains the choice of the fuzzy implication in (4): it has to be the S-implication based on the triangular norm used in (3), which will be denoted by* $\rightarrow_T$. E^Z and E_Z are fuzzy modifiers satisfying the following proposition.

Proposition 2. For F in $\mathcal{P}(U)$, we have:

$E_Z(F) \subseteq F \subseteq E^Z(F)$.

The proof is straightforward.

3 Proximity-Based Tolerant Inclusion

The basic idea is to introduce a certain tolerance into the inclusion indicator by taking into account the proximity between the elements of the domain considered. This can be done by replacing $A \subseteq B$ either by:

$A \subseteq_Z^1 B \equiv E_Z(A) \subseteq E^Z(B)$ or by:

$A \subseteq_Z^2 B \equiv (E_Z(A) \subseteq B \vee A \subseteq E^Z(B))$ or by:

$A \subseteq_Z^3 B \equiv (E_Z(A) \subseteq B \wedge A \subseteq E^Z(B))$.

Remark 1. One has:

$A \subseteq_Z^3 B \Rightarrow A \subseteq_Z^2 B \Rightarrow A \subseteq_Z^1 B$.

In the following, we focus on indicator $\subseteq_Z^3$, which acts both on A and B, and which is the most drastic way (among the three pointed out above) of relaxing the inclusion. We use the notation:

$$A \subseteq_Z B \Leftrightarrow (E_Z(A) \subseteq B \wedge A \subseteq E^Z(B)). \quad (5)$$

Remark 2. One could also consider two other relaxations, namely

$A \subseteq_Z^4 B \equiv E_Z(A) \subseteq B$ and

$A \subseteq_Z^5 B \equiv A \subseteq E^Z(B)$,

but it seems intuitively more reasonable, in general, to apply a relaxation mechanism that acts on both arguments.

To sum up, the principle is to say that A is included (with tolerance) in B iff *very(A)* is included in B and A is included in *more-or-less(B)* where the linguistic modifiers *very* and *more-or-less* are based on the notion of proximity.

3.1 Tolerant Inclusion of Crisp Sets

In the case where crisp sets are dealt with, one must use a Boolean proximity relation $E[Z]$ based on a regular interval $Z = [-\Omega, \Omega]$ centered in 0. One gets:

$E[Z](u, v)$ is true if $|u - v| \leq \Omega$, false otherwise.

Formulas (3) and (4) rewrite:

$$E^Z(F) \equiv \{s \in U \mid \exists r \in U \text{ such that } r \in F \wedge E[Z](r, s)\} \quad (6)$$

$$E_Z(F) \equiv \{s \in F \mid \forall r \in U, E[Z](r, s) \Rightarrow r \in F\} \quad (7)$$

Example 2. Let us consider the sets

$A = \{41, 59\}$ and $B = \{40, 48, 60\}$

defined on the interval [0, 100] of the integers, and the interval $Z = [-1, 1]$. We get:

$E_Z(A) = \varnothing$ and

$E^Z(B) = \{39, 40, 41, 47, 48, 49, 59, 60, 61\}$

and we have: $A \subseteq_Z B$.♦

3.2 Tolerant Inclusion of Fuzzy Sets

Here, the calculus – illustrated by the following example – is based on formulas (3) and (4).

Example 3. Let us consider the fuzzy sets:

$A = \{0.7/47, 0.9/48, 0.6/49, 1/50\}$

$B = \{0.6/41, 0.7/48, 1/49\}$

defined on the interval [0, 100] of the integers, and the fuzzy set Z represented by the t.m.f. (1, –1, 2, 2). According to (2), A is not included in B.

Using the triangular norm minimum in (3), and thus Kleene-Dienes implication in (4), we get:

$E_Z(A) = \{0.5/48, 0.5/49\}$ and

$E^Z(B) = \{0.5/39, 0.6/40, 0.6/41, 0.6/42, 0.5/43,$

0.5/46, 0.7/47, 1/48, 1/49, 1/50, 0.5/51}

$A \subseteq_Z B = (E_Z(A) \subseteq B)) \wedge (A \subseteq E^Z(B))) =$

$true \wedge true = true.$ ♦

3.3 Properties

Let us recall the three axioms valid for Boolean inclusion:

$A \subseteq B \Leftrightarrow B_c \subseteq A_c$ (A1)

where A_c (resp. B_c) denotes the complement of A (resp. B) in the universe U;

$A \subseteq (B \cap C) \Leftrightarrow (A \subseteq B) \wedge (A \subseteq C)$ (A2)

$A \subseteq B \Leftrightarrow S(A) \subseteq S(B)$ (A3)

where the set $S(A)$ is defined as $S(A)(x) = A(S(x))$ with a one-to-one mapping $s: X \rightarrow X$.

Let us now check whether these axioms remain valid when the regular inclusion is replaced by a tolerant one.

First case: crisp sets.

Axiom (A1). Do we have: $A \subseteq_Z B \Leftrightarrow B_c \subseteq_Z A_c$?

First, let us show that for a crisp set X, one has: $(E^Z(X))_c = E_Z(X_c)$.

Proof.

$(E^Z(X))_c(s) =$
$U - \{s \in U \mid \exists r \in U \text{ s.t. } r \in X \wedge E[Z](r, s)\}$
$= \{s \in U \mid \forall r \in U, r \notin X \vee \neg E[Z](r, s)\}$
$= \{s \in U \mid \forall r \in U, E[Z](r, s) \Rightarrow r \notin X\}$
$= E_Z(X_c)$.•

From this, it is straightforward to deduce that $(E_Z(X))_c = E^Z(X_c)$. Therefore, we have:

$A \subseteq_Z B \Leftrightarrow$

$(E_Z(A) \subseteq B \wedge A \subseteq E^Z(B)) \Leftrightarrow$

$B_c \subseteq (E_Z(A))_c \wedge (E^Z(B))_c \subseteq A_c \Leftrightarrow$

$B_c \subseteq E^Z(A_c) \wedge E_Z(B_c) \subseteq A_c \Leftrightarrow$

$B_c \subseteq_Z A_c$.

Axiom (A2). Do we have:

$A \subseteq_Z (B \cap C) \Leftrightarrow (A \subseteq_Z B) \wedge (A \subseteq_Z C)$?

First, let us notice that for two crisp sets B and C, one has: $E^Z(B \cap C) = E^Z(B) \cap E^Z(C)$ (the proof is straightforward). Therefore, we have:

$A \subseteq_Z (B \cap C) \Leftrightarrow$
$E_Z(A) \subseteq (B \cap C) \wedge A \subseteq E^Z(B \cap C) \Leftrightarrow$
$E_Z(A) \subseteq B \wedge E_Z(A) \subseteq C \wedge A \subseteq E^Z(B)$
$\wedge A \subseteq E^Z(C) \Leftrightarrow$

$(A \subseteq_Z B) \wedge (A \subseteq_Z C)$.

As to axiom(A3), it straightforwardly holds when $\subseteq$ is replaced by $\subseteq_Z$.

Second case: fuzzy sets.

Axiom (A1). Let us prove that this axiom is preserved if the fuzzy implication used in (4) is the S-implication based on the triangular norm used in (3). Let us recall that an S-implication $\rightarrow_\top$ is associated with a triangular norm $\top$ by the following relation:

$$a \rightarrow_\top b = 1 - \top(a, 1 - b). \quad (8)$$

Thus we have:

$E^Z(X_c)(s) = sup_{r \in U} \top(E[Z](s, r), 1 - X(r))$
$= sup_{r \in U} (1 - [E[Z](s, r) \rightarrow_T X(r))])$
$= 1 - inf_{r \in U} (E[Z](s, r) \rightarrow_T X(r))$
$= 1 - E_Z(X)(s)$
$= (E_Z(X))_c(s)$.

From this, it is straightforward to deduce that:

$(E^Z(X))_c = E_Z(X_c)$.

It follows that axiom (A1) holds (the proof is the same as in the crisp set case).

Axiom (A2). Only a weakened form of it holds when $\subseteq$ is replaced by $\subseteq_Z$ (the equivalence is replaced by an implication). First let us show that: $E^Z(B \cap C) \neq E^Z(B) \cap E^Z(C)$ in general.

Let $U = \{10, 11, 12\}$, $B = \{0.4/10, 0.7/11\}$, and $C = \{0.3/11, 0.8/12\}$. Let Z be represented by the t.m.f. (0, 0, 2, 2). Let us assume in this example that the intersection is interpreted by a minimum.

$B \cap C = \{0.3/11\}$

$E^Z(B \cap C) = \{0.3/10, 0.3/11, 0.3/12\}$

$E^Z(B) = \{0.5/10, 0.7/11, 0.5/12\}$

$E^Z(C) = \{0.3/10, 0.5/11, 0.8/12\}$

$E^Z(B) \cap E^Z(C) = \{0.3/10, 0.5/11, 0.5/12\}$.

Now, let us show that:

$E^Z(B \cap C) \subseteq E^Z(B) \cap E^Z(C)$.

Proof.

$\mu_{EZ(B \cap C)}(x) = sup_{y \in U} \top(E[Z](x, y), \mu_{B \cap C}(y))$

$= sup_{y \in U} \top(E[Z](x, y), \top(\mu_B(y), \mu_C(y)))$

$\leq \top(sup_{y \in U} \top(E[Z](x, y), \mu_B(y)),$

$$sup_{y \in U} \top(E[Z](x, y), \mu_C(y)))$$

$$\leq \top(\mu_{E^Z}(B)(x), \mu_{E^Z}(C)(x)).\bullet$$

This result entails that:

$A \subseteq_Z (B \cap C) \Rightarrow (A \subseteq_Z B) \wedge (A \subseteq_Z C)$.

The proof is the same as in the crisp set case, except that in the second line the equivalence must be replaced by an implication.

Remark. Dubois and Prade [10] have shown that S-implications and R-implications could be merged into a single family, provided that the class of triangular norms is enlarged to non-commutative conjunction operators. One has:

$$a \rightarrow b = 1 - cnj(a, 1 - b)$$

where *cnj* is a a triangular norm $\top$ if $\rightarrow$ is an S-implication, and *cnj* is a non-commutative conjunction *cnc* if $\rightarrow$ is an R-implication. This provides another way of defining E^Z and E_Z: one could use a non-commutative conjunction in (3) and the corresponding R-implication in (4), which would also preserve axiom (A1).

4. Application to the division of relations

4.1. Reminder About the Division

The relational division, i.e., the division of relation *r* of schema *R(A, X)* by relation *s* of schema *S(B)* (where *A* and *B* are compatible subsets of attributes) is defined as follows:

$$div(r, s, A, B) = \{x \mid s \subseteq \Omega(x)\} \quad (9)$$

where $\Omega(x) = \{a \mid <x, a> \in r\}$. In other words, an element *x* belongs to the result of the division if it is associated with *at least all* the values *a* of *B* appearing in *s*.

Example 4. Consider the relations suppliers (*s*) and census (c) of respective schemas *S(#store, chain, zipc, turnover)* and *C(city, zipc, pop)*. The query: “find the chains which have a store with a turnover greater than 0.5 k€ in every city from relation *c* whose population is over 200,000” can be formulated using a division:

div(proj(select(s, turnover > 0.5), {chain, zipc}), proj(select(c, pop > 200,000), {zipc}), {zipc}, {zipc})

where *select(r, cond)* is the selection of relation *r* with the condition cond and *proj(r, X)* is the projection of *r* onto the set of attributes *X*.

s

#store	chain	zipc	turnover
15	32	75000	1.2
12	32	69000	0.54
34	32	22000	0.25
26	32	13000	0.89
26	7	49000	0.37
78	7	35000	0.51

c

#city	zipc	pop
Paris	75000	2 125 800
Lyon	69000	445 274
Saint-Brieuc	22000	46 089
Marseille	13000	797 491
Angers	49000	151 322
Rennes	35000	206 194

Using the extensions of *s* and *c* above, one gets the result made of the single value <32>.♦

4.2. Tolerant Division of Crisp Relations

The tolerant inclusion introduced before can serve as a basis to define a tolerant division where the notion of proximity between values is taken into account. For instance, in the previous example, it might be useful to consider that often the stores are not in the big cities themselves but in their suburbs. Then, the zip codes in relation Supplier may be different from the ones associated with the different urban centers. Therefore, it is interesting to allow a certain tolerance on the zip code in the processing of the division (it is assumed that the difference between the zip codes reflects the distances between the cities). For instance, one may use a tolerance indicator such as $Z = (-900, 900, 0, 0)$. The tolerant division is interpreted by means of formula (9) where the regular inclusion is replaced by a tolerant one – which is itself interpreted using formulas (5) and (6).

Example 5. Let us consider again the context of example 4 and let us consider the extension of relation s below. For the chain 32, it can easily be checked that the set A = {75000, 69000, 13000, 35000} is included (with tolerance) in B = {75015, 69215, 22300, 13625}. Thus the result of the tolerant division is {32} whereas it would be empty if the regular division were used.

s	#store	chain	zipc	turnover
	15	32	75015	1.2
	12	32	69215	0.54
	34	32	22300	0.25
	26	32	13625	0.89
	26	7	49000	0.37
	78	7	35830	0.51

♦

4.3. Tolerant Division of Fuzzy Relations

Now, one considers fuzzy relations, i.e., relations obtained by applying flexible conditions on relations pertaining to a regular relational database. In this context, one can envisage queries similar to that of example 4, for instance the request: "find every chain *x*, to what extent is it true that *x* has a store with an important turnover in every largely populated city present in relation *c*". This query can be algebraically expressed thanks to a division of fuzzy relations as:

div(proj(select(s, turnover is important),
{chain, zipc}),
proj(select(c, pop is large), {zipc}),
{zipc}, {zipc}).

A fuzzy relation is defined as a fuzzy subset of the Cartesian product of domains. So, a fuzzy relation r is a set of weighted tuples, denoted by μ/t where μ is the membership degree of t in r, i.e., the extent to which t complies with the concept associated with r. In the context of the previous query, the degree tied to each tuple of relation *s' = select(s, turnover is important)* expresses the extent to which it is a store with an important turnover. A flexible query is made of operations applying to fuzzy relations and whose result is a fuzzy relation as well. Such operators are obtained through a natural extension of the usual algebraic operators (see, e.g., [5, 2]).

A straightforward (Boolean) extension of the division operator to fuzzy relations can be based on formula (9) where the inclusion is defined by formula (2). As to the *tolerant* division of fuzzy relations, it consists, as in the Boolean relation case, in replacing the inclusion present in formula (9) by a tolerant one. Then, the calculus of the tolerant inclusion is based on formulas (3) and (4).

5. Related works

As mentioned in the introduction, two other approaches have been previously proposed in order to relax the inclusion by tolerating exceptions.

The operator introduced in [3], called quantitative approximate inclusion, aims at authorizing, in the evaluation of $E \subseteq F$, a certain proportion of exceptions, i.e., of elements from *E* which are not in *F* (in the sense of a fuzzy implication which can be Rescher-Gaines'one when crisp sets are dealt with). Obviously, the principle of that approach is totally different from the one described here since tolerance in the proximity-based inclusion we propose is not a matter of a *number* of exceptions.

As to the operator introduced in [4], it aims at relaxing the family of graded inclusion indicators based on R-implications so as to authorize "low intensity" exceptions. Let us recall that R-implications can be expressed:

$p \rightarrow_{\text{R-i}} q = 1$ if $p \leq q, f(p, q)$ otherwise

where $f(p, q)$ expresses a degree of satisfaction of the implication when the antecedent p exceeds the conclusion q. The implications of Gödel ($p \rightarrow_{\text{Gö}} q = 1$ if $p \leq q$, q otherwise), Goguen ($p \rightarrow_{\text{Gg}} q = 1$ if $p \leq q$, q/p otherwise) and Lukasiewicz ($p \rightarrow_{\text{Lu}} q = 1$ if $p \leq q$, $1 - p + q$ otherwise) are the three most used R-implications. The principle of the so-called qualitative approximate inclusion is to introduce a tolerance on the implication values, that takes into account the gap betwen the membership degrees to *E* and to *F* respectively, for a given element *x*. More precisely, it is considered that the situation where $\mu_E(x)$ exceeds $\mu_F(x)$ is more or less acceptable as long as the difference $\mu_E(x) - \mu_F(x)$ is in a given interval.

The proximity-based tolerant inclusion proposed here can be called qualitative too, but in a different way than that outlined above. In [4], an exception *x* is ignored depending on the *membership degrees* attached to *x* in *E* and *F* respectively whereas in the proximity-based approach it will be ignored depending on the *values* present in the sets (and on the tolerance indicator *Z*). A consequence of this difference is that the proximity-based approach can apply to crisp sets while the qualitative approximate inclusion described in [4] cannot. On the other hand, the proximity-based approach in its form described here imposes that the domain be numeric. Nevertheless, in case of a non-numeric domain U, the approach can still be applied providing that a distance d be defined over U Then, E(u, v) can be defined as Z(d(u, v)) instead of Z(u – v).

As to proximity-based modifiers, the most similar work to our proposal is the recent one done by De Cock and Kerre in [8]. In that work, the authors introduce the two following fuzzy modifiers:

$$R^{\clubsuit}(A)(y) = \sup_{x \in X} \mathcal{C}(R(x, y), A(x)),$$

$$R^{\nabla}(A)(y) = \inf_{x \in X} \mathcal{T}(R(x, y), A(x)),$$

where *A* is a fuzzy set on the universe *X*, $\mathcal{C}$ is a conjunctor, $\mathcal{T}$ an implicator and *R(x, y)* a fuzzy relation. It has been shown that weakening and intensifying linguistic hedges can be modeled by such fuzzy modifiers. To do so, the authors make use of a particular suitable fuzzy relation. It is expressed by a *pseudo-metric based resemblance relation* that is reflexive and symmetric.

To provide some comparison between our approach and the one proposed by De Cock and Kerre, let us first emphasize that the motivations of these two approaches are quite different. Our starting point was mainly guided by the question: how can an *upper* and a *lower* approximation of a fuzzy set *F* be established using a parameterized proximity relation? It is that question that triggered the generalization of the use of a proximity relation to model fuzzy modifiers. Moreover, we can claim that our approach avoids the burden of the computation complexity since the calculus is reduced to simple fuzzy arithmetic operations. Let us also mention that the proximity relation-based approach to fuzzy modifiers seems more flexible from a user point of view. For instance, it allows for preserving either the core or the support if needed, when modifying a fuzzy set.

6. Conclusion

Various extensions of set inclusion have been proposed in the framework of fuzzy sets. In this paper, the novelty is to consider an proximity-based inclusion indicator, in order to take into account the closeness between the elements of the domain. Such an operator, based on a tolerance indicator defined over the domain considered, has been defined and its practical utility has been illustrated by the relational division of fuzzy relations. Let us notice that, beyond the database domain, the division operation plays a central role in information retrieval (and in web search engines) where the issue is to retrieve the documents that "include" all the concepts corresponding to a given set of keywords. The tolerant division we propose could thus enable to relax in an explicit way the requirement about the keywords that must appear in a document to make it more or less acceptable, providing that a semantic distance has been defined over the terms that may appear in the queries and the documents.

The tolerant inclusion considered in this paper is Boolean-valued, and one of the perspectives of this work is to extend it in order to define a graded tolerant inclusion of fuzzy sets. Then, it would be of interest to check whether the graded inclusion obtained satisfies the axioms pointed out in [7]. It would also be worth studying whether the quotient property of the division operator is preserved when one moves to a tolerant division such as that introduced in section 4.

References

[1] W. Bandler, L. Kohout, Fuzzy power sets and fuzzy implication operators, Fuzzy Sets and Systems 4 (1980) 13-30.

[2] P. Bosc, B. Buckles, F. Petry, O. Pivert, Fuzzy Databases, in: Fuzzy Sets in Approximate Reasoning and Information Systems, J. Bezdek, D. Dubois and H. Prade (Eds.), The Handbook of Fuzzy Sets Series, Kluwer Academic Publishers, 1999, pp. 403-468.

[3] P. Bosc, O. Pivert, About approximate inclusion and its axiomatization, Fuzzy Sets and Systems 157 (2006) 1438-1454.

[4] P. Bosc, O. Pivert, On a qualitative approximate inclusion – Application to the division of fuzzy relations, Proc. of the Int. Workshop on Flexible Database and Information System Technology (FlexDBIST'06), in conjunction with *DEXA'06*, Cracow, Poland, September 6, 430-434, 2006.

[5] P. Bosc, D. Dubois, O. Pivert, H. Prade, Flexible queries in relational databases – the example of the division operator, Theoretical Computer Science 171 (1997) 281-302.

[6] P. Bosc, A. Hadjali, O. Pivert, Modelling fuzzy modifiers by means of a proximity relation, Proc. of the 25th International Conference of the North American Fuzzy Information Processing

Society (NAFIPS'06), Montréal, Canada, June 3-6, 2006.

[7] C. Cornelis, C. Van der Donck, E. Kerre, Sinha-Dougherty approach to the fuzzification of set inclusion revisited, Fuzzy Sets and Systems 134 (2003) 283-296.

[8] M. De Cock, E. Kerre, Fuzzy modifiers based on fuzzy relations, Information Sciences, 160 (2004) 173-199.

[9] D. Dubois, A. HadjAli, H. Prade, Fuzzy qualitative reasoning with words, in: Computing with Words (P.P. Wang, Ed.), Vol. 3, John Wiley & Son, 2001, pp. 347-366.

[10] D. Dubois and H. Prade, A theorem on implication functions defined from triangular norms, Stochastica, vol. 8, pp. 267-279, 1984. Also in: Readings in Fuzzy Sets for Intelligent Systems (Dubois D., Prade H., Yager R.R. eds.), Morgan & Kaufmann, 1993, pp. 105-112.

[11] L.A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 371-384.