Concept lattices under non-commutative conjunctors are generalized concept lattices

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Abstract

Generalized concept lattices have been recently proposed to deal with uncertainty or incomplete information as a non-symmetric generalization of the theory of fuzzy formal concept analysis. On the other hand, concept lattices have been defined as well in the framework of fuzzy logics with non-commutative conjunctors.

The contribution of this paper is to prove that any concept lattice for non-commutative fuzzy logic can be interpreted inside the framewok of generalized concept lattices, specifically, it is isomorphic to a sublattice of the cartesian product of two generalized concepts lattices.

Keywords : formal concept analysis, concept lattices, Galois connections.

1 Introduction

The theory of Formal Concept Analysis has its beginnings in the works of Ganter and Wille [5] where an object-attribute view of data is developed. Specifically, a concept is defined as a pair of subsets which, respectively mean the extension (the subset of objects related to the concept) and the intension (the set of attributes which define the concept).

Ganter-Wille's approach was based on a classical setting, in that objects (resp. attributes) either crisply belong or not to the extension (resp. intension) of a concept. Since then, there have been several approaches aiming at introducing some kind of fuzziness, vagueness or uncertainty in the data. Fuzzy concept lattices were firstly introduced by Burusco and Fuentes-González in [4], later independently developed by Pollandt in [11] and Bělohlávek in [2] (which also considered fuzzy orderings). Later, Georgescu and Popescu [6] defined the notion of fuzzy concept lattice associated to fuzzy logic with a non-commutative conjunction. More recently, Krajči considered the so-called generalized concept lattices, which use different sets of truth-values to refer to the subset of objects, to the subset of attributes of a concept, as well as to the degree to which an object has an attribute.

The framework of generalized concept lattices is actually a generalization of Pollandt's concept lattice and Bělohlávek's concept lattice (when considering a classical ordering) and, furthermore, it has been shown to embed some other approaches, like the concept lattices with hedges [7].

In this paper, the frameworks of Krajči's generalized concept lattice and Georgescu-Popescu's construction for non-commutative conjunctions are related; specifically, we show the relationship between the residuated operators introduced by the latter with the notion of left-continuity introduced by the former. This relationship allows to interpret Georgescu-Popescu concept lattices as a particular case of the generalized concept lattices.

2 Generalized Concept Lattices

As stated in the introduction, generalized concept lattices are based on two complete lattices $(L_1, \preceq_1)$ $(L_2, \preceq_2)$, and a poset $(P, \leq)$, which are the different sets of truth-values to refer to the objects, to the attributes of a concept, as well as to the degree to which an object has an attribute.

In addition, a conjunction operator $\otimes\colon L_1 \times L_2 \to P$ is considered, which is assumed to be increasing and left-continuous in both arguments. The notion of left-continuity, see [8], is given below:

Definition 2.1 *Given* $(L, \preceq)$ *a complete lattice and*

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$(P, \leq)$ *a poset, an application* $T\colon L \to P$ *is* left-continuous *when, given* $p \in P$ *and a non-empty subset* $X \subseteq L$*, the following condition holds:*

> *if* $T(x) \leq p$ *for every* $x \in X$*, then* $T(\sup X) \leq p$

The *context* under which concepts are defined is a tuple $(A, B, R, \otimes)$, where the sets A and B represent the attributes and objects, and $R\colon A \times B \to P$ is a P-fuzzy relation.

Given a context $(A, B, R, \otimes)$, two maps ${}^{\uparrow}\colon {L_2}^B \to {L_1}^A$ and ${}^{\downarrow}\colon {L_1}^A \to {L_2}^B$ can be defined as follows:

$$\begin{aligned} g^{\uparrow}(a) &= \sup\{x \in L_1 \mid (\forall b \in B)\ x \otimes g(b) \leq R(a,b)\} \\ f^{\downarrow}(b) &= \sup\{y \in L_2 \mid (\forall a \in A) f(a) \otimes y \leq R(a,b)\} \end{aligned}$$

Now, the concepts given by the context are defined as the elements (g, f) of ${L_2}^B \times {L_1}^A$ such that $g^{\uparrow} = f$ and $f^{\downarrow} = g$.

By relying on the fact that the pair $({}^{\downarrow}, {}^{\uparrow})$ is a Galois connection, Krajči proved that the set of concepts

$$\mathcal{G} = \{(g, f) \mid g^{\uparrow} = f \text{ and } f^{\downarrow} = g\}$$

is actually a complete lattice with the ordering $\preceq_{\mathcal{G}}$ defined as

$(g_1, f_1) \preceq_{\mathcal{G}} (g_2, f_2)$ if and only if $g_1 \preceq_2 g_2$

The lattice $\mathcal{G}$ is known as the *generalized concept lattice* associated to $(A, B, R, \otimes)$.

3 Concept lattice for non-commutative conjunctors

Georgescu and Popescu introduced in [6] a concept lattice which is based on the structure of complete biresiduated lattice[1] as underlying set for the truth-values of both the objects and attributes. The formal definition of this structure is given below:

Definition 3.1 *A* complete biresiduated lattice *is a tuple* $(L, \preceq, \&, \swarrow, \nwarrow)$ *satisfying the following conditions:*

1. $(L, \preceq)$ *is a complete lattice.*
2. $(L, \&, \top)$ *is a monoid.*
3. *The adjoint properties:*
   (a) $x \preceq z \swarrow y$ *if and only if* $x \,\&\, y \preceq z$
   (b) $y \preceq z \nwarrow x$ *if and only if* $x \,\&\, y \preceq z$

Note that the adjoint conditions are different precisely when the conjunctor $\&$ is non-commutative.

The study of implications and conjunctions related by adjointness has recently been the subject of extensive research, becoming an important branch of the theory of multiple-valued and fuzzy logic. Note that this structure was introduced in the framework of fuzzy logic programming in [10] and, simultaneously, under the name of implication triple, in [1].

In order to define the concept lattice, we have to introduce the notion of context. Given a complete biresiduated lattice $(L, \preceq, \&, \swarrow, \nwarrow)$, a (biresiduated) context is a tuple (A, B, R) where A, B are sets representing the attributes and the objects, respectively, and $R\colon A \times B \to L$ is a L-fuzzy relation.

Now, given a context (A, B, R) the mappings ${}^{\upuparrows}$, ${}^{\Uparrow}\colon L^B \to L^A$ and ${}^{\downdownarrows}$, ${}^{\Downarrow}\colon L^A \to L^B$ are defined as follows:

$$\begin{aligned} g^{\upuparrows}(a) &= \inf\{R(a,b) \swarrow g(b) \mid b \in B\} \\ g^{\Uparrow}(a) &= \inf\{R(a,b) \nwarrow g(b) \mid b \in B\} \\ f^{\Downarrow}(b) &= \inf\{R(a,b) \nwarrow f(a) \mid a \in A\} \\ f^{\downdownarrows}(b) &= \inf\{R(a,b) \swarrow f(a) \mid a \in A\} \end{aligned}$$

The concepts in this framework are triples $(g, f, f') \in L^{A \times B \times B}$ such that $g^{\upuparrows} = f$; $g^{\Uparrow} = f'$; $f^{\Downarrow} = g$; ${f'}^{\downdownarrows} = g$; this is why we will call them *t-concepts*.

The fact that the pairs $({}^{\upuparrows}, {}^{\Downarrow})$ and $({}^{\Uparrow}, {}^{\downdownarrows})$ form Galois connections is used in [6] to prove that the set of t-concepts, denoted as $\mathcal{L}$, is a complete lattice under the ordering $(g_1, f_1, {f'}_1) \preceq_{\mathcal{L}} (g_2, f_2, {f'}_2)$ if and only if $g_1 \preceq g_2$ (equivalently $f_2 \preceq f_1$ or ${f'}_2 \preceq {f'}_1$).

4 Relating both frameworks

In order to embed the concept lattice $\mathcal{L}$ into Krajči's framework, firstly we have to know when the operator $\&$, defined in Section 3, is left continuous in both arguments.

By definition, we have that $\&$ has associated two "residuated" applications $\swarrow$ and $\nwarrow$ satisfying the adjoint properties. As a result we obtain that $\&$ is sup-preserving in both arguments, i. e., for all $x, y \in L$ and $X, Y \subseteq L$ we have that $\sup(X) \,\&\, y = \sup\{x' \,\&\, y \mid x' \in X\}$, and $x \,\&\, \sup(Y) = \sup\{x \,\&\, y' \mid y' \in Y\}$, see [1].

Once we know that $\&$ is sup-preserving in both arguments, the following step is to obtain left-continuity. This can be achieved as an application of the following result which characterises when an operator is sup-preserving in terms of the left-continuity.

Lemma 4.1 *Let* $(L, \preceq)$ *be a complete lattice and*

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[1] The term used by Georgescu-Popescu is complete generalized residuated lattice, which we do not use here in order to avoid misunderstandings with Krajči's.

$\wedge\colon L \times L \to L$ *an increasing operator then the following conditions are equivalent:*

1. *$\wedge$ is sup-preserving in the first argument.*
2. *$\wedge$ is left-continuous in the first argument and $\bot \wedge y = \bot$ for every $y \in L$.*

Proof: (1 implies 2)

The proof of the boundary condition is trivial considering $X = \varnothing$ since $\bot \wedge y = \sup(X) \wedge y = \sup\{x \wedge y \mid x \in X\} = \bot$. Now, given y, $z \in L$ and a non-empty subset $X \subseteq L$, if $x \wedge y \preceq z$ for every $x \in X$ then $\sup\{x \wedge y \mid x \in X\} \preceq z$, so, by hypothesis:

$$\sup(X) \wedge y = \sup\{x \wedge y \mid x \in X\} \preceq z$$

therefore $\wedge$ is left-continuous in the first argument.

(2 implies 1)

Let $\varnothing \neq X \subseteq L$ and $y \in L$, the inequality $\sup\{x \wedge y \mid x \in X\} \preceq \sup(X) \wedge y$ follows directly from the increasing character of $\wedge$ and definition of supremum.

For the other inequality, since $x \wedge y \preceq \sup\{x \wedge y \mid x \in X\}$ for every $x \in X$, we can use the left-continuity in the first argument of $\wedge$, and obtain $\sup(X) \wedge y \preceq \sup\{x \wedge y \mid x \in X\}$.

If $X = \varnothing$, the equality is straightforward because of the boundary condition and $\sup(X) = \bot$. ■

A similar lemma can be proved for the second argument, but in this case the boundary condition has to be modified as $x \wedge \bot = \bot$. As a consequence of Lemma 4.1, we get that $\&$ is left continuous in both arguments.

Remark 4.1 *Note that only the first implication is needed for our purposes. However, we have stated and proved the full equivalence in order to point out a gap in [3], in which there is no explicit mention to the need of the boundary conditions, which turns out to be essential.*

Finally, an alternative definition of the Galois connections of Section 3 can be given in terms of suprema, hence obtaining a definition more similar to that of Krajči's:

Lemma 4.2 *Given a complete biresiduated lattice $(L, \preceq, \&, \swarrow, \nwarrow)$ and a biresiduated context (A, B, R) we have that:*

$$\begin{aligned} g^{\upuparrows}(a) &= \sup\{x \in L_1 \mid (\forall b \in B) x \,\&\, g(b) \preceq R(a,b)\} \\ f^{\Downarrow}(b) &= \sup\{y \in L_2 \mid (\forall a \in A) f(a) \,\&\, y \preceq R(a,b)\} \end{aligned}$$

Proof: For the first equality we need to prove that

$$\sup\{x \in L_1 \mid (\forall b \in B) x \,\&\, g(b) \preceq R(a,b)\} = \inf\{R(a,b) \swarrow g(b) \mid b \in B\}$$

By the adjoint property of $\&$ with respect to $\swarrow$, and the characterisation of the infimum as the supremum of the lower bounds, we obtain

$$\begin{aligned} \sup\{x \in L \mid (\forall b \in B) x \,\&\, g(b) \preceq R(a,b)\} &= \\ = \sup\{x \in L \mid (\forall b \in B) x \preceq R(a,b) \swarrow g(b)\} \\ = \inf\{R(a,b) \swarrow g(b) \mid b \in B\} \end{aligned}$$

The other equality is proved in a similar way. ■

Theorem 4.1 *Given a complete biresiduated lattice $(L, \preceq, \&, \swarrow, \nwarrow)$ and a biresiduated context (A, B, R), then there exist two generalized concept lattices, $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the sublattice of $\mathcal{G}_1 \times \mathcal{G}_2$ defined by*

$$\mathcal{G}_{12} = \{((g_1, f_1), (g_2, f_2)) \in \mathcal{G}_1 \times \mathcal{G}_2 \mid g_1 = g_2\}$$

is isomorphic to the lattice of t-concepts $\mathcal{L}$.

Proof: By Lemma 4.2, we have that the definitions of the applications $({}^{\upuparrows}, {}^{\Downarrow})$ given in Section 3 coincide with the definitions of $({}^{\uparrow}, {}^{\downarrow})$ given in Section 2 considering the context $(A, B, R, \&)$.

Now, if we consider the operator $\&^{op}\colon L \times L \to L$, where $x \,\&^{op}\, y = y \,\&\, x$, we obtain similarly that the pair $({}^{\Uparrow}, {}^{\downdownarrows})$ is equal to $({}^{\uparrow^{op}}, {}^{\downarrow_{op}})$ defined for the context $(A, B, R, \&^{op})$.

Finally, we simply have to take $\mathcal{G}_1$ and $\mathcal{G}_2$ as the generalized concept lattices associated to the contexts $(A, B, R, \&)$ and $(A, B, R, \&^{op})$ ■

5 Conclusions

We have shown how the framework of concept lattices for non-commutative fuzzy logics introduced by Georgescu and Popescu naturally embeds in the framework of generalized concept lattices.

As future work we are planning to study the relationship between the generalized concept lattice and the recently introduced multi-adjoint concept lattice [9].

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