A Fuzzy Representation for Weights of Alternatives in AHP

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Abstract

AHP (Analytic Hierarchy Process) has been widely used in decision making techniques. However, the results from AHP often lose reliability because the comparison matrix does not always have sufficient consistency. In these cases, fuzzy representation for weighting activities using results from a sensitivity analysis is useful. In this paper, we present alternative overall weights by employing some assumptions. Since an idea of less ambiguity is employed, the results show how AHP has fuzziness when the comparison matrix is not sufficiently consistent. An example is also shown in this paper.

Keywords: Decision making, AHP (Analytic Hierarchy Process), Fuzzy sets, Sensitivity analysis, Weights representation.

1 Introduction

AHP (Analytic Hierarchy Process) was proposed by Saaty T.L. in 1977 [1], [2]. The method has been popular and widely used in the domain of decision making, since it can include vagaries such as humans feelings. In addition, it can be developed to ANP (Analytic Network Process) models. However, the comparison matrix often does not have enough consistency when AHP is used since, for instance, a problem may contain too many activities for decision making. In these cases, we consider that answers from decision-makers (i.e. components of the comparison matrix) have ambiguity or fuzziness. For resolving this type of problem, fuzzy reciprocal components have been proposed as components of the data matrix in some research [10]. In this paper, we consider that weights should also have ambiguity or fuzziness. Therefore, it is necessary to represent these weights by use of fuzzy sets.

Sensitivity analysis is applied to AHP to analyze the amount the components of a pairwise comparison matrix influences the weights and consistency of a matrix. This makes it possible to show the magnitude of the fuzziness in the weights.

In previous research, we proposed a new representation for weights of activities [7][8]. For this study, a developed representation of weights of alternatives is proposed. It is represented as L-R fuzzy numbers by using the results from the sensitivity analysis. This paper encompasses methodology to represent weights by fuzzy sets. In addition, a representation of fuzziness as a result of AHP is presented when a comparison matrix does not have enough consistency.

2 Summary of AHP

2.1 Process of AHP

(Process 1) Representation of structure by a hierarchy. The problem under consideration can be represented in a hierarchical structure. The highest level of the hierarchy consists of a unique element that is the overall objective. At the lower levels, there are multiple activities (i.e. elements within a single level) with relationships among elements of the adjacent higher level to be considered. The activities are evaluated using subjective judgments of a decision maker. Elements that lie at the upper level are called parent elements while those that lie at lower level are called child elements. Alternative elements are put at the lowest level of the hierarchy

(Process 2) Paired comparison between elements at each level. A pairwise comparison matrix A is created from a decision maker's answers. Let n be the number of elements at a certain level. The upper triangular components of the comparison matrix a_{ij} ($i< j = 1,\dots,n$) are 9, 8, .. , 2, 1, 1/2, …, or 1/9. These denote intensities of importance from activity i to j. The lower triangular components a_{ji} are described with reciprocal numbers as follows

$$a_{ji} = 1/a_{ij}\ , \tag{1}$$

in addition, for diagonal elements, let $a_{ii} = 1$. The lower triangular components and diagonal elements are occasionally omitted from the written equation as they are evident if upper triangular components are shown. The decision maker should make $n(n-1)/2$ paired comparisons at a level with n elements.

(Process 3) Calculations of weight at each level. The weights of the elements, which represent grade of importance among each element, are calculated from the pairwise comparison matrix. The eigenvector that corresponds to a positive eigenvalue of the matrix is used in calculations throughout in this paper.

(Process 4) Priority of an alternative by a composition of weights. The composite weight can be calculated from the weights of one level lower. With repetition, the weights of the alternative, which are the priorities of the alternatives with respect to the overall objective, are finally found.

2.2 Consistency index

Since components of the comparison matrix are obtained by comparisons between two elements, coherent consistency is not guaranteed. In AHP, the consistency of the comparison matrix A is measured by the following consistency index (C.I.)

$$\text{C.I.} = \frac{\lambda_A - n}{n-1}, \tag{2}$$

where n is the order of matrix A, and λ_A is its maximum eigenvalue.

It should be noted that $\text{C.I.} \geq 0$ holds. And if the value of C.I. becomes smaller, then the degree of consistency becomes higher, and vice versa. The comparison matrix is consistent if the following inequality holds.

$$\text{C.I.} \leq 0.1 \tag{3}$$

3 Sensitivity Analysis of AHP

When AHP is used, the comparison matrix is often inconsistent or large differences among the overall weights of the alternatives do not appear. Thus, it is very important to investigate how the components of a pairwise comparison matrix influence the consistency or weights. Sensitivity analysis is used to analyze how results are influenced when certain variables change. Therefore, it is necessary to establish a sensitivity analysis of AHP.

In our research, a previously proposed method [7] is used to evaluate the fluctuation of the consistency index and weights when a comparison matrix is perturbed. This method is useful as it does not change the structure of the data.

Evaluating the consistency index and the weights of a perturbed comparison matrix are performed as follows.

(1) Perturbations $\varepsilon a_{ij} d_{ij}$ are imparted to component a_{ij} of a comparison matrix, and the fluctuation of the consistency index and the weight are expressed by the power series of ε.

(2) Fluctuations of the consistency index and the weights are represented by the linear combination of d_{ij}.

(3) By the coefficient of d_{ij}, it can be shown that how the component of the comparison matrix gives influence on the consistency index and the weight.

Since the pairwise comparison matrix A is a positive square matrix, the following Perron-Frobenius theorem [4] holds.

Theorem 1 (Perron – Frobenius) *For a positive square matrix A, the following holds true.*

1. *Matrix A has a positive eigenvalue. If λ_A is the largest eigenvalue then λ_A is a simple root. The positive eigenvector* **w***, corresponding to λ_A, exists. λ_A is called the Frobenius root of A.*
2. *Any positive eigenvectors of A are the constant multiples of* **w***.*
3. *The absolute value of the eigenvalues of A, except for λ_A, is smaller than λ_A.*
4. *The Frobenius root of the transposed matrix A' is equivalent to the Frobenius root of A.*

This theorem ensures the existence of a weight vector in a pairwise comparison matrix.

From Theorem 1, the following theorem regarding a perturbed comparison matrix holds true [7].

Theorem2 *Let* $A = (a_{ij})$*,* $i,j = 1,\ldots,n$ *be a comparison matrix and let* $A(\varepsilon) = A+\varepsilon D_A$*,* $D_A=(a_{ij}d_{ij})$ *be a matrix that has been perturbed. Moreover, let* λ_A *be the Frobenius root of A with* $\boldsymbol{w}_1$ *being the corresponding eigenvector. Let* $\boldsymbol{w}_2$ *be the eigenvector corresponding to the Frobenius root of A', then, the Frobenius root* $\lambda(\varepsilon)$ *of* $A(\varepsilon)$ *and the corresponding eigenvector* $\boldsymbol{w}_1(\varepsilon)$ *can be expressed as follows*

$$\lambda(\varepsilon) = \lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon), \tag{4}$$

$$\boldsymbol{w}_1(\varepsilon) = \boldsymbol{w}_1 + \varepsilon\boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon), \tag{5}$$

where

$$\lambda^{(1)} = \frac{\boldsymbol{w}_2' D_A \boldsymbol{w}_1}{\boldsymbol{w}_2' \boldsymbol{w}_1}, \tag{6}$$

$\boldsymbol{w}^{(1)}$ *is an n-dimension vector that satisfies*

$$(A - \lambda_A I)\boldsymbol{w}^{(1)} = -(D_A - \lambda^{(1)} I)\boldsymbol{w}_1, \tag{7}$$

where $\boldsymbol{o}(\varepsilon)$ *denotes an n-dimension vector in which all components are* $o(\varepsilon)$*.*

Proof of this theorem can be found in Ohnishi's paper [7].

3.1 Sensitivity analysis of the consistency index

Regarding a fluctuation of the consistency index, the following corollary can be obtained from Theorem 2.

Corollary 1 *Using an appropriate* g_{ij}*, we can represent the consistency index* C.I.(ε) *of the perturbed comparison matrix as follows*

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon\sum_i^n\sum_j^n g_{ij} d_{ij} + o(\varepsilon). \tag{8}$$

(Proof)

From the definition of the consistency index (3) and (4),

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon\frac{\lambda^{(1)}}{n-1} + o(\varepsilon).$$

Let $\boldsymbol{w}_1=(w_{1i})$ and $\boldsymbol{w}_2=(w_{2i})$ from (6). $\lambda^{(1)}$ is can now be represented as

$$\lambda^{(1)} = \frac{1}{\boldsymbol{w}_2' \boldsymbol{w}_1}\sum_i^n\sum_j^n w_{2i} a_{ij} w_{1j} d_{ij},$$

therefore, the second part of the right side is expressed by a linear combination of d_{ij}. (Q.E.D)

g_{ij} in equation (8) in Corollary 1 shows the influence of comparison matrix components on the consistency.

On the other hand, since the comparison matrix $A(\varepsilon) = (a_{ij}(\varepsilon))$ is reciprocal, then $a_{ji}(\varepsilon) = 1/a_{ij}(\varepsilon)$ and becomes

$$a_{ji} + \varepsilon a_{ji} d_{ji} = \frac{1}{a_{ij}} - \varepsilon\frac{d_{ij}}{a_{ij}} + o(\varepsilon). \tag{9}$$

Here, since $a_{ji} = 1/a_{ij}$,

$$d_{ji} = -d_{ij} \tag{10}$$

is obtained. The impact on the consistency can be easily shown by use of this property.

3.2 Sensitivity analysis of weights

With regards to the fluctuation in weighs, the following corollary can also be obtained from Theorem 2.

Corollary 2 *Using an appropriate* $h_{ij}^{(k)}$*, we can represent the fluctuation* $\boldsymbol{w}^{(1)}=(w_k^{(1)})$ *of the weight* (*i.e. the eigenvector corresponding to the Frobenius root*) *as follows*

$$w_k^{(1)} = \sum_i^n\sum_j^n h_{ij}^{(k)} d_{ij}. \tag{11}$$

(Proof)

The k-th row component of the right side of (7) in Theorem 2 is represented as

$$\sum_i^n\sum_j^n\left\{\frac{w_{1k} w_{2i} a_{ij} w_{1j}}{\boldsymbol{w}_2' \boldsymbol{w}_1} - \delta(i,k) a_{ij} w_{1j}\right\} d_{ij},$$

and is expressed by a linear combination of d_{ij}. Here,$\delta(i,k)$ is Kronecker's symbol

$$\delta(i,k) = \begin{cases} 1 & (i = k), \\ 0 & (i \neq k). \end{cases}$$

In contrast, since λ_A is a simple root, Rank(A-$\lambda_A I$) = n-1. Accordingly, the weight vector is normalized as

$$\sum_k^n (w_{1k} + \varepsilon w_k^{(1)}) = \sum_k^n w_{1k} = 1,$$

then the condition is as follows.

$$\sum_{k}^{n} w_k^{(1)} = 0. \tag{12}$$

By using an elementary transformation to formula (7) in the condition above, we also can represent $w_k^{(1)}$ by linear combinations of d_{ij}. (Q.E.D)

As seen in equation (5) in Theorem 2, the component that has a great influence on weight $\boldsymbol{w}_1(\varepsilon)$ is the component which has the greatest influence on $\boldsymbol{w}^{(1)}$. $h_{ij}^{(k)}$ in equation (11) from Corollary 2 shows how the influence by the components of a comparison matrix on the weights can be calculated.

The influence can also be shown easily by use of equation (10).

4 Representation of weights using fuzzy sets

The comparison matrix often has poor consistency (i.e. 0.1<C.I.<0.2) because it encompasses several activities. In these cases, the components of a comparison matrix are considered to have fuzziness since they result from the fuzzy judgment of humans. Therefore, weights should be treated as fuzzy numbers.

To represent fuzziness of weight w_{1k}, an L-R fuzzy number is used.

4.1 L-R fuzzy number

L-R fuzzy number

$$M = (m, \alpha, \beta)_{LR}$$

is defined as fuzzy sets whose membership function is as follows.

$$\mu_M(x) = \begin{cases} R\left(\dfrac{x-m}{\beta}\right) & (x > m), \\ L\left(\dfrac{m-x}{\alpha}\right) & (x \le m). \end{cases}$$

where $L(x)$ and $R(x)$ are shape function which satisfies

(1) $L(x) = L(-x)$,

(2) $L(0) = 1$,

(3) $L(x)$ is a non increasing function

4.2 Fuzzy weights of activities

From the fluctuation of the consistency index, the multiple coefficient $g_{ij}h_{ij}^{(k)}$ in Corollary 1 and 2 is considered as the influence on a_{ij}.

Since g_{ij} is always positive, if the coefficient $h_{ij}^{(k)}$ is positive, the real weight of activity k is considered to be larger than w_{1k}. Conversely, if $h_{ij}^{(k)}$ is negative, the real weight of activity k is considered to be smaller. Therefore, the sign of $h_{ij}^{(k)}$ represents the direction of the fuzzy number spread. The absolute value $g_{ij}|h_{ij}^{(k)}|$ represents the size of the influence.

On the other hand, if C.I. becomes bigger, then the judgment becomes more fuzzy.

Consequently, multiple C.I. $g_{ij}|h_{ij}^{(k)}|$ can be regarded as a spread of a fuzzy weight $\tilde{w}_k$ concerned with a_{ij}.

Definition 1 (fuzzy weight) *Let w_{1k} be a crisp weight of activity k, and g_{ij} $|h_{ij}^{(k)}|$ denote the coefficients found in Corollary 1 and 2. If 0.1<C.I.<0.2, then a fuzzy weight $\tilde{w}_k$ is defined by*

$$\tilde{w}_k = (w_{1k}, \alpha_k, \beta_k)_{LR} \tag{13}$$

where

$$\alpha_k = \text{C.I.}\sum_{i}^{n}\sum_{j}^{n} s(-, h_{kij}) g_{ij} \mid h_{kij} \mid, \tag{14}$$

$$\beta_k = \text{C.I.}\sum_{i}^{n}\sum_{j}^{n} s(+, h_{kij}) g_{ij} \mid h_{kij} \mid, \tag{15}$$

$$s(+,h) = \begin{cases} 1, (h \ge 0) \\ 0.(h < 0) \end{cases}, \quad s(-,h) = \begin{cases} 1, (h < 0) \\ 0.(h \ge 0) \end{cases}$$

4.2 Fuzzy weights of alternatives

Using the fuzzy weights of activities defined above and local crisp weights of alternatives with respect to certain activities, we can calculate overall weights from the viewpoint of the overall objective by extension. However, the results from the operation of fuzzy numbers are frequently too ambiguous to interpret.

Fuzzy weights of activities are normalized thus their sum is 1, therefore we can avoid much ambiguity since this condition has been considered [9].

In general, operating with constraints is difficult but can be accomplished if every fuzzy membership function is linear.

Especially for every normal triangular function with a core u_i, the constraint $\sum_i^n u_i = 1$ holds, and the order of singleton coefficients is assumed. Thus, the upper and lower limit of α -cut sets of linear sum can be easily calculated.

Let $f_t(x_k)$ be a crisp local weight of alternative *t* with respect to activity *k,* and in this paper, assume $0 \le f_t(x_1) \le f_t(x_2) \le \cdots \le f_t(x_n)$. Then, the overall weight of an alternative *t* is also the L-R fuzzy number and is represented as follows.

$$\tilde{v}_t = (v_t, l_t, r_t)_{LR}$$

where

$$v_t = \sum_k^n w_{1k} f_t(x_k),$$

$$l_t = v_t - \inf \mathrm{supp}(\tilde{v}_t), \quad r_t = \sup \mathrm{supp}(\tilde{v}_t) - v_t$$

In the above equations, inf supp, sup supp are lower and upper limits of support sets and are calculated as follows.

$$\inf \mathrm{supp}(\tilde{v}_t) = \max_j \left[\sum_{i=1}^{j-1} (w_{1i} + \beta_i) f_t(x_i) + \sum_{i=j+1}^{n} (w_{1i} - \alpha_i) f_t(x_i) + \left\{ 1 - \sum_{i=1}^{j-1} (w_{1i} + \beta_i) - \sum_{i=j+1}^{n} (w_{1i} - \alpha_i) \right\} f_t(x_j) \right]$$

$$\sup \mathrm{supp}(\tilde{v}_t) = \min_j \left[\sum_{i=1}^{j-1} (w_{1i} - \alpha_i) f_t(x_i) + \sum_{i=j+1}^{n} (w_{1i} + \beta_i) f_t(x_i) + \left\{ 1 - \sum_{i=1}^{j-1} (w_{1i} - \alpha_i) - \sum_{i=j+1}^{n} (w_{1i} + \beta_i) \right\} f_t(x_j) \right]$$

5 Example

In this section, we introduce an example of an employment selection problem, whose structure is shown in Figure 1.

Table 1 shows a comparison matrix of activities, and Table 2 shows weights of alternatives with respect to activities.

Calculated weights of activities are as follows.

$$\begin{pmatrix} \text{Salary} \\ \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Holiday} \\ \text{Welfare} \end{pmatrix} = \begin{pmatrix} 0.04 \\ 0.41 \\ 0.26 \\ 0.16 \\ 0.04 \\ 0.09 \end{pmatrix}$$

However, they do not have much reliability, because consistency of the matrix is not so good (C.I.=0.13).

Table 3 shows a result of sensitivity analysis of consistency. The comparison value between salary and stability, or the value between holiday and welfare has much influence.

Table 4 shows fuzzy weights of activities. A weight of stability has most fuzziness.

Table 1: A comparison matrix of activities

	Salary	Stability	Prospect	Attractive	Holiday	Welfare
Salary	1	1/5	1/5	1/5	1/2	1/3
Stability		1	3	4	7	5
Prospect			1	3	6	5
Attractive				1	7	3
Holiday					1	1/5
Welfare						1

Table 2: Weights of alternatives with respect to activities.

	Company A	Company B	Company C
Salary	0.158	0.766	0.076
Stability	0.042	0.180	0.778
Prospect	0.180	0.778	0.042
Attractive	0.070	0.751	0.178
Holiday	0.157	0.249	0.594
Welfare	0.121	0.115	0.764

Table 3: Result of sensitivity analysis of consistency

	Salary	Stability	Prospect	Attractive	Holiday	Welfare
Salary	0.0000	0.3602	0.2291	0.1414	0.0847	0.1279
Stability	0.0789	0.0000	0.3254	0.2677	0.1123	0.1817
Prospect	0.1199	0.0864	0.0000	0.3052	0.1463	0.2761
Attractive	0.1907	0.1031	0.0874	0.0000	0.2715	0.2635
Holiday	0.3394	0.2620	0.1945	0.1028	0.0000	0.0782
Welfare	0.2131	0.1536	0.0977	0.1005	0.3613	0.0000

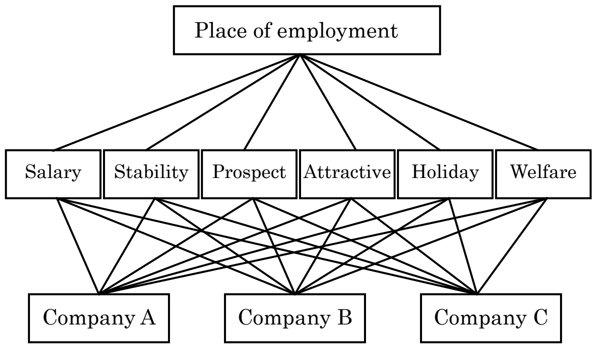

Figure 1: A hierarchical structure

Table4: fuzzy weights of activities

	Center	Spread(L)	Spread(R)
Salary	0.0381	0.0011	0.0012
Stability	0.4118	0.0085	0.0089
Prospect	0.2620	0.0059	0.0059
Attractive	0.1616	0.0026	0.0040
Holiday	0.0387	0.0013	0.0013
Welfare	0.0877	0.0022	0.0023

Finally we can find the overall alternative weights (Table5).

Table5: fuzzy weights of alternatives

	Center	Spread(L)	Spread(R)
Company A	0.0983	0.0306	0.0306
Company B	0.4484	0.1476	0.1659
Company C	0.4532	0.1872	0.1723

6. Conclusions

We proposed a representation for the overall weights of alternatives by use of fuzzy sets and the result of a sensitivity analysis for cases in which consistency of the comparison matrix is poor. Our approach shows how to represent weights, as well as how the result of AHP has fuzziness, when inconsistency exists. This was due to reduced ambiguity in the representation presented in this work compared to previous normal fuzzy operations. Moreover we show an example of the alternative fuzzy weights.

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