Ordinal Means

Anna Kolesárová
Institute IAM, FCHFT
Slovak University of Technology
Radlinského 9, 812 37 Bratislava
Slovakia
anna.kolesarova@stuba.sk

Gaspar Mayor
Department of Mathematics
and Computer Science
University of the Balearic Islands
Palma de Mallorca
Spain
gmayor@uib.es

Radko Mesiar
Department of Mathematics
and Descriptive Geometry, FCE
Slovak University of Technology
Radlinského 11, 813 68 Bratislava
Slovakia
and
ÚTIA AV ČR, 182 08 Prague,
Czech Republik
mesiar@math.sk

Abstract

The aim of the contribution is the discussion of some types and classes of means on ordinal scales, especially kernel and shift invariant ordinal means, weighted ordinal means based on weighted divisible t–conorms (t–norms) and dissimilarity based ordinal means. Moreover, several types of ordinal arithmetic means are introduced.

Keywords: Ordinal aggregation operator, Ordinal mean, Divisible t–conorm, Dissimilarity function.

1 Introduction

Typical means on the cardinal scale $[0, 1]$ are the arithmetic mean M and several of its generalizations, such as quasi–arithmetic means, weighted arithmetic means, weighted quasi–arithmetic means, OWA operators, weighted ordered quasi–arithmetic means. Recall that OWA operators can be viewed as weighted arithmetic means applied not directly to given input values, but to the ordered ones. Weighted ordered quasi–arithmetic means are defined in a similar way. Moreover, the class of weighted quasi–arithmetic means contains M, and also quasi–arithmetic and weighted arithmetic means.

All till now mentioned means are characterized by weights of single inputs (possibly reordered) and by a transformation function (generator). Several other types of means on $[0, 1]$ are characterized by certain special properties. For example, kernel aggregation operators are means with Chebyshev norm equal to 1 [4, 14, 15]. Shift invariant aggregation operators are means commuting with (acceptable) shifts [24, 29]. Comonotone additivity is the property characterizing means related to the Choquet integral, while max- and min–homogenity characterize the Sugeno integral–based means [6, 28].

In [3] Calvo and Mesiar introduced weighted (continuous) t–conorms, and their class of aggregation functions also covers all weighted quasi–arithmetic means for which 0 is not an annihilator. Observe that by duality weighted (continuous) t–norms can be introduced, covering all weighted quasi–arithmetic means for which the element 1 is not an annihilator, and thus the union of both these classes contains all quasi–arithmetic means.

Ordinal scales become more and more important, especially because of the "computerization" of several branches of human thinking [8, 22, 10, 19, 23]. Thus, practical applications of fuzzy logic are limited to a finite number of truth values. Firstly, technical implementations allow us to work only with a finite (though very large) number of values. Secondly, when representing vagueness, it is usually meaningless to distinguish a high number of truth values; only a small number suffices. Also note that the reasoning with linguistic truth degrees reduces, from a mathematical point of view, to processing on a fixed discrete ordinal scale related to the number of different truth degrees involved.

The aim of this contribution is the discussion of some types and classes of means on ordinal scales (ordinal means, for short). The paper is organized as follows. In the next section kernel and shift invariant ordinal means are defined and some of their properties are studied. In Section 3, the notions of weighted divisible t–conorms and t–norms on the discrete scale are introduced and next exploited for defining the lower and upper ordinal arithmetic (weighted arithmetic) means. Section 4 is devoted to the dissimilarity based ordinal means. Finally, in Conclusion a generalization of the results obtained in Sections 3 and 4 for arithmetic ordinal means is proposed.

2 Kernel and shift invariant ordinal means

Each finite scale with $m+1$ elements can be represented by the scale $L_m = \{0, 1, \ldots, m\}$.

Definition 1 *Let $n \in \mathbb{N}$. A non–decreasing mapping $A : L_m^n \to L_m$ is called an n–ary ordinal mean whenever*

$$\min(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n) \tag{1}$$

for all $(x_1, \ldots, x_n) \in L_m^n$. An ordinal mean is a mapping $A : \bigcup_{n\in\mathbb{N}} L_m^n \to L_m$ such that $A_{|L_m^n}$ is an n–ary ordinal mean for each $n \in \mathbb{N}$.

Note that due to the monotonicity of ordinal means, the boundary condition (1) can be replaced by idempotency:

$$A(x, \ldots, x) = x \quad \text{for all } x \in L_m. \tag{2}$$

Definition 2 *An n–ary ordinal mean $A : L_m^n \to L_m$ is an n–ary kernel ordinal mean if for all $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in L_m^n$*

$$\begin{aligned} |A(x_1, \ldots, x_n) &- A(y_1, \ldots, y_n)| \\ &\leq \max(|x_1 - y_1|, \ldots, |x_n - y_n|). \end{aligned} \tag{3}$$

A mapping $A : \bigcup_{n\in\mathbb{N}} L_m^n \to L_m$ is a kernel ordinal mean if $A_{|L_m^n}$ is an n–ary kernel ordinal mean for each $n \in \mathbb{N}$.

Definition 3 *An n–ary ordinal mean $A : L_m^n \to L_m$ is a shift invariant n–ary ordinal mean if*

$$A(x_1 + a, \ldots, x_n + a) = a + A(x_1, \ldots, x_n) \tag{4}$$

for each $a \in \{1, \ldots, m\}$ and all $x_1, \ldots, x_n \in \{0, 1, \ldots, m - a\}$. A mapping $A : \bigcup_{n\in\mathbb{N}} L_m^n \to L_m$ is a shift invariant ordinal mean if $A_{|L_m^n}$ is a shift invariant n–ary ordinal mean for each $n \in \mathbb{N}$.

For n–ary means shift invariantness is equivalent to the requirement $A(x_1+1, \ldots, x_n+1) = 1 + A(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in (L_m \setminus \{m\})^n$.

Following the ideas of the corresponding result for kernel aggregation operators in [4], it can be shown that an ordinal mean A is a kernel ordinal mean if and only if it is sub–shift invariant, i.e.,

$$A(x_1 + 1, \ldots, x_n + 1) \leq 1 + A(x_1, \ldots, x_n) \tag{5}$$

for all $(x_1, \ldots, x_n) \in (L_m \setminus \{m\})^n$.

In decision making an important and desirable property of aggregation operators is their *joint strict monotonicity* which in the case of ordinal means can be characterized by the inequality

$$A(x_1, \ldots, x_n) < A(x_1 + 1, \ldots, x_n + 1) \tag{6}$$

for all $(x_1, \ldots, x_n) \in (L_m \setminus \{m\})^n$ and for each $n \in \mathbb{N}$.

Note that for each shift invariant ordinal mean $A : L_m^n \to L_m$ and all $(x_1, \ldots, x_n) \in L_m^n$ it holds

$$A(x_1, \ldots, x_n) = a + A(x_1 - a, \ldots, x_n - a),$$

where $a = \min(x_1, \ldots, x_n)$. Thus to know A it is enough to know its values at points $(x_1, \ldots, x_n) \in L_m^n$ such that $0 \in \{x_1, \ldots, x_n\}$, i.e., $(x_1, \ldots, x_n) \in L_m^n \setminus (L_m \setminus \{0\})^n = L_{m,0}^n$. Vice–versa, each non–decreasing mapping $B : L_{m,0}^n \to L_m$ bounded from above by the max–operator can be extended to a shift invariant mapping $A_B : L_m^n \to L_m$ defined by

$$A_B(x_1, \ldots, x_n) = a + B(x_1 - a, \ldots, x_n - a), \tag{7}$$

where $a = \min\{x_1, \ldots, x_n\}$. Unfortunately, A_B need not be monotone, thus not a mean. Based on the results from [24, 29] and their proofs, we can derive the next representation.

Proposition 1 *A mapping $A_B : L_m^n \to L_m$ given by (7) is a shift invariant ordinal mean if and only if B possesses the zero–kernel property*

$$\begin{aligned} |B(x_1, \ldots, x_n) &- B(y_1, \ldots, y_n)| \\ &\leq \max(|x_1 - y_1|, \ldots, |x_n - y_n|) \end{aligned}$$

for all $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in L_{m,0}^n$ such that $x_i = y_i = 0$ for some $i \in \{1, \ldots, n\}$.

It is evident that each jointly strictly monotone ordinal mapping $A : L_m^n \to L_m$ which is non–decreasing, is an n–ary ordinal mean. Mordelová and Muel proved [27] that each binary jointly strictly monotone kernel ordinal mean is exactly a shift invariant ordinal mean. This result can easily be extended for arbitrary $n \in \mathbb{N}$.

Proposition 2 *Let $n \in \mathbb{N}$. For a mapping $A : L_m^n \to L_m$ the following claims are equivalent*

(i) *A is an n–ary shift invariant ordinal mean.*

(ii) *A is an n–ary jointly strictly monotone kernel ordinal mean.*

Proof. In the light of Proposition 1, the proof that (i) $\Rightarrow$ (ii) is trivial. Conversely, suppose that A is an n–ary jointly strictly monotone kernel ordinal mean. Due to the kernel property of A, for each $(x_1, \ldots, x_n) \in (L_m \setminus \{m\})^n$ it holds $A(x_1 +$

$1, \ldots, x_n + 1) - A(x_1, \ldots, x_n) \in \{0, 1\}$. The joint strictly monotonicity of A ensures $A(x_1+1, \ldots, x_n+1) - A(x_1, \ldots, x_n) > 0$, and thus $A(x_1+1, \ldots, x_n+1) - A(x_1, \ldots, x_n) = 1$, which implies the shift invariantness of A.

□

3 Weighted ordinal means

In this section we will present ordinal means corresponding to weighted (quasi-)arithmetic means, OWA–operators and weighted ordered quasi–arithmetic means. They are based on the original ideas of Godo and Torra [7] exploiting ordinal divisible t–conorms (t–norms) on L_m which were modified in [17] in such a way that the procedure always results in an ordinal mean. We introduce the formula for weighted ordinal means in an equivalent form based on the special representation of divisible ordinal t–conorms.

Recall that continuous t–conorms on the scale $[0, 1]$ were characterized as follows (for more details we refer to [1, 30, 18, 12]).

Proposition 3 *A function $S : [0, 1]^2 \to [0, 1]$ is a continuous t–conorm if and only if, there is a finite or countably infinite set K, a family $(\,]a_k, b_k[\,)_{k \in K}$ of non–empty, parwise disjoint open subintervals of $[0, 1]$, and a family $(g_k)_{k \in K}$, $g_k : [a_k, b_k] \to [0, \infty]$, of continuous, strictly increasing functions with $g_k(a_k) = 0$, for each $k \in K$, such that*

$$S(x, y) = \begin{cases} g_k^{(-1)}\left(g_k(x) + g_k(y)\right) & \\ \qquad \text{if } (x, y) \in\,]a_k, b_k]^2, & \\ \max\{x, y\} \qquad \text{otherwise}, & \end{cases} \tag{8}$$

where $g_k^{(-1)} : [0, \infty] \to [a_k, b_k]$ is the pseudo–inverse of g_k, see [13], given by

$$g_k^{(-1)}(x) = \sup\{z \in [a_k, b_k] \mid g_k(z) \leq x\}.$$

The notion of weighted continuous t–conorms on $[0, 1]$ was introduced in [3].

Definition 4 *Let $S : [0, 1]^2 \to [0, 1]$ be a continuous t–conorm given by (8), and let $\mathbf{w} = (w_1, \ldots, w_n) \in [0, 1]^n$ be a normal weighting vector, i.e., with weights satisfying the property $\sum_{i=1}^{n} w_i = 1$. The weighted t–conorm $S_{\mathbf{w}} : [0, 1]^n \to [0, 1]$ is given by*

$$S_{\mathbf{w}}(x_1, \ldots, x_n) = \begin{cases} g_k^{(-1)}\left(\sum_{i=1}^{n} w_i\, g_k\left(\max\{a_k, x_i\}\right)\right) & \\ \qquad \text{if } \max\{x_i \mid w_i > 0\} \in\,]a_k, b_k], & \\ \max\{x_i \mid w_i > 0\} \qquad \text{otherwise}. & \end{cases} \tag{9}$$

Example 1

(i) Let $S_L : [0, 1]^2 \to [0, 1]$ be the Łukasiewicz t–conorm, $S_L(x, y) = \min\{x + y, 1\}$, i.e., $K = \{1\}$, $a_1 = 0$, $b_1 = 1$, $g_1 : [0, 1] \to [0, \infty]$ is given by $g_1(x) = x$. Then for the uniform weighting vector $\mathbf{w}_u = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ we have $(S_L)_{\mathbf{w}_u}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$, i.e., $(S_L)_{\mathbf{w}_u} = M$ is the arithmetic mean.

For a general normal weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$, i.e., $\mathbf{w} \in [0, 1]^n$, $\sum_{i=1}^{n} w_i = 1$, $(S_L)_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i\, x_i$, i.e., $(S_L)_{\mathbf{w}}$ is a weighted arithmetic mean. Note that each weighted arithmetic mean W can be represented in this way.

(ii) Let $S : [0, 1]^2 \to [0, 1]$ be a continuous Archimedean t–conorm, i.e.,

$$S(x, y) = g^{-1}(\min\{g(1), g(x) + g(y)\})$$

for some strictly increasing continuous function $g : [0, 1] \to [0, \infty]$ with $g(0) = 0$. Then for any normal weighting vector $\mathbf{w}$,

$$S_{\mathbf{w}}(x_1, \ldots, x_n) = g^{-1}\left(\sum_{i=1}^{n} w_i\, g(x_i)\right),$$

which means that $S_{\mathbf{w}}$ is a weighted quasi–arithmetic mean. It has annihilator 1 whenever $g(1) = \infty$ (i.e., when S is a strict t–conorm) and each weight w_i is positive. For $\mathbf{w} = \mathbf{w}_u$, $S_{\mathbf{w}_u}$ is a quasi–arithmetic mean.

Weighted continuous t–norms $T_{\mathbf{w}}$ are defined in a similar way as weighted t–conorms.

Observe that each t–conorm on L_m is continuous. However, the property equivalent to the continuity of t–conorms on $[0, 1]$ is the divisibility, see, e.g., [11], and thus in the framework of discrete t–conorms we will deal with their divisibility [8, 23]. Divisible t–conorms on L_m were characterized in [21]:

Proposition 4 *A function $S : L_m^2 \to L_m$ is a divisible t–conorm on L_m if and only if, there is a set $\{b_0, \ldots, b_j\} \subset L_m$, $b_0 = 0 < b_1 < \ldots < b_j = m$ such* *that*

$$S(x,y) = \begin{cases} \min\{b_k, x+y-b_{k-1}\} \\ \qquad\qquad\qquad \textit{if } (x,y) \in]b_{k-1}, b_k]^2, \\ \\ \max\{x,y\} \qquad\quad \textit{otherwise.} \end{cases} \tag{10}$$

Divisible t–norms on L_m are in a one–to–one correspondence with divisible t–conorms on L_m throughout Frank's functional equation

$$T(x,y) + S(x,y) = x + y. \tag{11}$$

Each divisible t–conorm S on L_m given by (10) can also be represented in form (9), putting $a_k = b_{k-1}$ and $g_k : \{a_k, a_k+1, \ldots, b_k\} \to [0,\infty]$ given by $g_k(x) = x - a_k$, and the pseudo–inverse $g_k^{(-1)} : [0,\infty] \to \{a_k, \ldots, b_k\}$ given by

$$g_k^{(-1)}(x) = \sup\{z \in \{a_k, \ldots, b_k\} \mid g_k(z) \le x\}.$$

Now, to define a divisible weighted t–conorm $S_{\mathbf{w}}$ on L_m, we can formally repeat Definition 4.

Definition 5 *Let $\mathbf{w} \in [0,1]^n$ be a normal weighting vector and let $S : L_m^2 \to L_m$ be a divisible t–conorm. The weighted divisible t–conorm $S_{\mathbf{w}} : L_m^n \to L_m$ is given by*

$$S_{\mathbf{w}}(x_1, \ldots, x_n) = \begin{cases} g_k^{(-1)}(w_i \max\{x_i, b_{k-1}\}) \\ \qquad\qquad \textit{if } \max\{x_i \mid w_i > 0\} \in]b_{k-1}, b_k], \\ \\ 0 \qquad\qquad \textit{otherwise.} \end{cases} \tag{12}$$

From equation (11) we can introduce weighted divisible t–norms on L_m, by

$$T_{\mathbf{w}}(x_1, \ldots, x_n) = \begin{cases} \inf\Big\{z \in \{b_{k-1}, \ldots, b_k\} \mid \\ \qquad\qquad z \ge \sum\limits_{i=1}^{n} w_i \min\{x_i, b_k\}\Big\} \\ \qquad\qquad \text{if } \min\{x_i \mid w_i > 0\} \in [b_{k-1}, b_k[, \\ \\ 1 \qquad\qquad \text{otherwise.} \end{cases} \tag{13}$$

The only divisible Archimedean t–conorm on L_m is the Łukasiewicz t-conorm, given by $S_L(x,y) = \min\{x+y, m\}$, and the corresponding weighted t–conorm for the uniform weighting vector $\mathbf{w}_u$, $(S_L)_{\mathbf{w}_u} : L_m^n \to L_m$, given by

$$(S_L)_{\mathbf{w}_u}(x_1, \ldots, x_n) = \sup\left\{z \in L_m \mid z \le \frac{1}{n}\sum_{i=1}^{n} x_i\right\} = \lfloor M(x_1, \ldots, x_n) \rfloor,$$

can be understood as the lower arithmetic mean on L_m (and denoted by M_L). Here, $\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor of a real x and M is the standard arithmetic mean on $\mathbb{R}$.

Similarly, $(T_L)_{\mathbf{w}_u} : L_m^n \to L_m$, defines the upper arithmetic mean on L_m,

$$(T_L)_{\mathbf{w}_u}(x_1, \ldots, x_n) = \inf\left\{z \in L_m \mid z \ge \frac{1}{n}\sum_{i=1}^{n} x_i\right\} = \lceil M(x_1, \ldots, x_n) \rceil$$

(it will be denoted by M_U), where $\lceil x \rceil : \mathbb{R} \to \mathbb{Z}$ is the ceiling of a real x.

The lower weighted arithmetic means on L_m can be introduced as $(S_L)_{\mathbf{w}}$, while the upper ones as $(T_L)_{\mathbf{w}}$,

$$(S_L)_{\mathbf{w}}(x_1, \ldots, x_n) = \lfloor W(x_1, \ldots, x_n) \rfloor,$$

$$(T_L)_{\mathbf{w}}(x_1, \ldots, x_n) = \lceil W(x_1, \ldots, x_n) \rceil.$$

Obviously, lower and upper ordinal OWA operators on L_m can be introduced by

$$(OWA_m)_* = \lfloor OWA \rfloor \text{ and } (OWA_m)^* = \lceil OWA \rceil.$$

Observe that formula (12) can also be written in the form

$$\lfloor W(\max\{x_1, b_{k-1}\}, \ldots, \max\{x_n, b_{k-1}\}) \rfloor,$$

whenever $\max\{x_i \mid w_i > 0\} \in]b_{k-1}, b_k]$.

Formulae (12) and (13) present ordinal forms of weighted divisible t–conorms and t–norms. The fact that there is a unique divisible Archimedean t–conorm (t–norm) on L_m, namely S_L (T_L), excludes the possibility of introducing proper quasi–arithmetic (weighted quasi–arithmetic) means on L_m using the above approach.

If we consider a general weighting vector $\mathbf{v} = (v_1, \ldots, v_n) \in [0,\infty[^n$, $\sum\limits_{i=1}^{n} v_i > 0$, we first normalize it, $\mathbf{w} = \frac{\mathbf{v}}{\sum_{i=1}^{n} v_i}$, and then we put $S_{\mathbf{v}} = S_{\mathbf{w}}$ ($T_{\mathbf{v}} = T_{\mathbf{w}}$).

4 Dissimilarity based ordinal means

Dissimilarity based means on real intervals were introduced and studied in [25], compare also [5]. Here we adopt one of the approaches discussed in the cited papers to the ordinal scales L_m.

Definition 6 *Let $K : \mathbb{R} \to \mathbb{R}$ be a convex function with unique minimum $K(0) = 0$ and let $f : L_m \to \mathbb{R}$ be a strictly monotone function. The function $D_{K,f} : L_m^2 \to \mathbb{R}$ given by*

$$D_{K,f}(i,j) = K(f(i) - f(j))$$

is called a dissimilarity function.

To define a symmetric ordinal mean exploiting a dissimilarity function $D_{K,f}$, one first needs to define a "middle point" of any interval $\{i, i+1, \ldots, j\}$, $0 \leq i \leq j \leq m$. Though there are several consistent possibilities, we will deal with two approaches only, namely, with the "lower middle point" $LMP(i,j) = \lfloor \frac{i+j}{2} \rfloor$ and the "upper middle point" $UMP(i,j) = \lceil \frac{i+j}{2} \rceil$.

Definition 7 *Let $D_{K,f}$ be a given dissimilarity function.*

(i) *The mapping $M_{K,f,L} : L_m^n \to L_m$ given by*

$$M_{K,f,L}(x_1, \ldots, x_n) = LMP(l, u) \tag{14}$$

where

$$l = \min\{k \in L_m \mid \sum_{i=1}^{n} K(f(x_i) - f(k)) = \min\}$$

and

$$u = \max\{k \in L_m \mid \sum_{i=1}^{n} K(f(x_i) - f(k)) = \min\}$$

is called a lower (K,f)–ordinal mean.

(ii) *Similarly, the mapping $M_{K,f,U} : L_m^n \to L_m$ given by*

$$M_{K,f,U}(x_1, \ldots, x_n) = UMP(l, u) \tag{15}$$

is called an upper (K,f)–ordinal mean

Proposition 5 *Both (K,f)–ordinal means defined in Definition 7 are symmetric ordinal means.*

If K is an even function, then

$$M_{K,id,U}(x_1, \ldots, x_n)$$
$$= m - M_{K,id,L}(m - x_1, \ldots, m - x_n).$$

Remark 1 Observe that for any dissimilarity function $D_{K,f}$ and weights $\mathbf{w} = (w_1, \ldots, w_n) > \mathbf{0}$ we can introduce a lower (an upper) (K,f)–ordinal mean minimizing the expression

$$\sum_{i=1}^{n} w_i \, K(f(x_i) - f(k))$$

in Definition 7.

Recall that for the function $K : \mathbb{R} \to \mathbb{R}$ given by $K(x) = |x|$ ($K = Abs$) and $f : L_m \to \mathbb{R}$ given by $f(i) = i$ ($f = id$), the corresponding (Abs, id)–ordinal means are ordinal medians which were discussed first in [20]. Note that for each odd m both, lower and upper ordinal medians coincide with the classical median on L_m. Observe that on any real interval I the (Abs, id)–mean is exactly the median operator, see [25, 5]. Similarly, putting $K = Q$, where $Q(x) = x^2$, $x \in \mathbb{R}$, (Q, id)–mean on any real interval I yields the standard arithmetic mean. Thus we can introduce the lower arithmetic mean on L_m as the lower (Q, id)–ordinal mean, and the upper arithmetic mean on L_m as the upper (Q, id)–ordinal mean. Due to Remark 1, weighted ordinal arithmetic means can be defined. Obviously, then also ordinal OWA's on L_m can be introduced. Coming back to a real interval I, (Q, f)–mean for a continuous and strictly monotone function $f : I \to \mathbb{R}$ yields a quasi–arithmetic mean on I generated by f. Thus (Q, f)–ordinal means on L_m can be understood as ordinal quasi–arithmetic means. Moreover, we can introduce weighted ordinal quasi–arithmetic means.

Example 2 Let $m = 3$, i.e., let us work on the scale $L_3 = \{0, 1, 2, 3\}$. Put $n = 4$ and consider the input $\mathbf{x} = (2, 0, 3, 0)$. Then

(i) If $K = Abs$, $f = id$, then $l = 0$, $u = 2$ and $M_{Abs,id,L}(\mathbf{x}) = M_{Abs,id,U}(\mathbf{x}) = 1$.

(ii) If $K = Q$, $f = id$, then $l = u = 1$ and $M_{Q,id,L}(\mathbf{x}) = M_{Q,id,U}(\mathbf{x}) = 1$.

(iii) If $K = Q$, $f = Q$, then $l = u = 2$ and $M_{Q,Q,L}(\mathbf{x}) = M_{Q,Q,U}(\mathbf{x}) = 2$.

(iv) If $K(x) = \begin{cases} x & \text{if } x \geq 0, \\ \frac{2}{7} x^2 & \text{if } x \leq 0 \end{cases}$, $f = id$, then $l = 1$, $u = 2$ and $M_{K,id,L}(\mathbf{x}) = 1$, $M_{K,id,U}(\mathbf{x}) = 2$.

Note that all dissimilarity based ordinal means, lower and upper ones, are shift invariant, i.e., jointly strictly monotone kernel ordinal means.

5 Conclusion

The lower ordinal arithmetic mean M_L derived in Section 3 is defined for all $\mathbf{x} = (x_1, \ldots, x_n) \in L_m^n$ by $M_L(\mathbf{x}) = \left\lfloor \sum_{i=1}^{n} \frac{x_i}{n} \right\rfloor$. Similarly, the upper ordinal arithmetic mean M_U is given by $M_U(\mathbf{x}) = \left\lceil \sum_{i=1}^{n} \frac{x_i}{n} \right\rceil$.

In Section 4, for dissimilarity based ordinal arithmetic means it holds

$$M_{Q,id,L}(\mathbf{x}) = \begin{cases} \left\lfloor \sum_{i=1}^{n} \frac{x_i}{n} \right\rfloor \\ \quad \text{if} \quad \sum_{i=1}^{n} \frac{x_i}{n} - \left\lfloor \sum_{i=1}^{n} \frac{x_i}{n} \right\rfloor \le \frac{1}{2}, \\ \left\lceil \sum_{i=1}^{n} \frac{x_i}{n} \right\rceil \quad \text{otherwise,} \end{cases}$$

and

$$M_{Q,id,U}(\mathbf{x}) = \begin{cases} \left\lfloor \sum_{i=1}^{n} \frac{x_i}{n} \right\rfloor \\ \quad \text{if} \quad \sum_{i=1}^{n} \frac{x_i}{n} - \left\lfloor \sum_{i=1}^{n} \frac{x_i}{n} \right\rfloor < \frac{1}{2}, \\ \left\lceil \sum_{i=1}^{n} \frac{x_i}{n} \right\rceil \quad \text{otherwise.} \end{cases}$$

These results lead to introducing two classes of ordinal arithmetic means, namely, $(M_{c,L})_{c\in[0,1[}$ and $(M_{c,U})_{c\in\,]0,1]}$, where

$$M_{c,L}(\mathbf{x}) = \begin{cases} i & \text{if} \quad i \le \sum_{i=1}^{n} \frac{x_i}{n} \le i + c, \\ i+1 & \text{if} \quad i + c < \sum_{i=1}^{n} \frac{x_i}{n} < i + 1, \end{cases}$$

$$M_{c,U}(\mathbf{x}) = \begin{cases} i & \text{if} \quad i \le \sum_{i=1}^{n} \frac{x_i}{n} < i + c, \\ i+1 & \text{if} \quad i + c \le \sum_{i=1}^{n} \frac{x_i}{n} < i + 1. \end{cases}$$

It holds:

$$\begin{aligned} M_L &= M_{1,U} \quad \text{and} \quad M_U = M_{0,L}, \\ M_{Q,id,L} &= M_{\frac{1}{2},L} \quad \text{and} \quad M_{Q,id,U} = M_{\frac{1}{2},U} \end{aligned}$$

and

$$M_{c,L}(x_1,\ldots,x_n) = m - M_{1-c,U}(m-x_1,\ldots,m-x_n)$$

for all $(x_1,\ldots,x_n) \in L_m^n$.

Evidently, all these ordinal means are symmetric, shift invariant, thus jointly strictly monotone kernel ordinal means on L_m

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