Modeling the context in Multi-Criteria Decision Analysis

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Abstract

In Multiple Criteria Decision Analysis, the preferences of the decision maker regarding each criterion are classically modelled by a utility function depending on one single variable representing the point of view attached to the criteria. There are situations in which the preferences regarding an attribute depend on more variables. In this case, we propose an extended model taking into account all contributing variables. The main asset of our approach is that it does not requires much more data from the decision maker than the classical model. We also propose an explanation framework.

Keywords: decision support systems, multiple criteria analysis, utility functions, context.

1 Introduction

Multi-Criteria Decision Analysis (MCDA) aims at helping a Decision Maker (DM) in selecting one option among several or in assessing several options, on the basis of a family $N = \{1, \ldots, n\}$ of viewpoints characterizing the various consequences of this selection or these assessments. An essential part of this analysis is the construction of a multi-criteria model deemed to represent the preferences $\succeq$ of the DM. For two options a and b, $a \succeq b$ means that a is at least as good as b to the DM. We consider here the transitive decomposable model [7]

$$a \succeq b \iff H(u_1(a), \ldots, u_n(a)) \geq H(u_1(b), \ldots, u_n(b))$$

where $u_k(a)$ measures the satisfaction of the DM regarding option a considering only viewpoint $k \in N$, and H is an aggregation function.

MCDA is very often applied to problems where the options of interest $A = \{a^1, \ldots, a^m\}$ are either known in advance or constructed during the early stage of the analysis. In this case, the multi-criteria model is usually specified only on the options of A. In particular, utility function u_k is constructed only on the elements of A, and thus $u_k : A \to \mathbb{R}$.

There are many applications where an explicit representation of the utility functions on the set of options is not possible. Let us describe three examples. In the first example, the number of options is either infinite or very large so that it is not possible to elicit the satisfaction degree u_k of the DM explicitly on all options. In the two other examples, the multi-criteria model must be constructed without a knowledge of the options on which it will be applied later. The second example corresponds to bid evaluation in public call for tenders [1, 2]. The model must thus be specified before the competition begins, and must be made public to the potential candidates. It defines the *rules* under which the candidates will be assessed. In the last example, the multi-criteria model is implemented in decision support system as an embedded function that has to provide an assessment each time a new situation arises. One can mention automatic assessment of trainees in a simulator or automatic situation assessment in a surveillance system.

These applications are very different from previous case since the model shall produce meaningful comparisons or assessments on all potential options that can arise. All these options may cover very different situations that have to be anticipated beforehand. The model that is constructed must thus be generic enough to take all these situations into consideration.

In previous examples, the consequence of potential options on each viewpoint $k \in N$ must necessarily be described by a set X_k called *descriptor* or *attribute*. This set corresponds to a mono-dimensional variable that is the indicator or metric that is used to represent the viewpoint. An option a is thus described by a value $a_k \in X_k$ on each attribute, and utility function u_k is now a function from X_k onto $\mathbb{R}$.

The representation of a viewpoint $k \in N$ by one mono-dimensional attribute X_k is not always satisfactory. It happens indeed that a same value of an attribute X_k corresponds to different situations and is interpreted differently by the DM. Let us give an example. Consider a training simulator in which trainees practice shooting on targets with a riffle. The main viewpoint to be considering when evaluating the quality of a shot is the sighting shift which is the distance between the point that the trainee has aimed with the riffle and the center of the target. This variable corresponds to the descriptor associated with the main viewpoint. The smaller this value the better. However, a value of, say $10\,cm$ is not always judged in the same manner by the instructor. If the target is at short range and is not moving, $10\,cm$ is judged bad, whereas if the target is at long range and is moving, this same figure is judged very good. This means that in order to assess attribute *sighting shift*, one needs to know the distance of the target as well as its speed. These two variables characterize the context under which the attribute must be analyzed.

More generally, an attribute X_k must sometimes be analyzed with other variables called hereafter *contextual variables* described the sets $Y_k^1, \ldots, Y_k^{p_k}$. Let $P_k = \{1, \ldots, p_k\}$ be the contextual variables for viewpoint k. We want to stress that the contextual variables do not measure the consequence on another viewpoint. The utility function u_k are thus defined as $u_k : X_k \times Y_k^1 \times \cdots \times Y_k^{p_k} \to \mathbb{R}$.

From now on, we consider the case where set X_k is continuous. We propose in this paper a method to construct a utility function $u_k : X_k \times Y_k^1 \times \cdots \times Y_k^{p_k} \to \mathbb{R}$, requiring a number of data provided by the DM which is not much more than what is done for usual mono-dimensional utility functions.

The rest of the paper is organized as follows. Section 2 describes the Macbeth approach that aims in specifying mono-dimensional utility functions. Section 3 shows some naive ways to tackle the multidimensional aspect of utility functions and gives their drawbacks. The model of representation of the multi-dimensional utility functions is explained in Section 4. We explain the data that must be provided by the DM to specify the model. The interpolation method that enables the DM to compute the utility value once the model has been specified is given in Section 5. The model that is proposed in this paper is a little bit more complex than the mono-dimensional one for a user. It becomes thus necessary to help the DM in understanding the result of the evaluation by proposing an automatic generation of an explanation in a Decision Support System. This is precisely the goal of Section 6.

2 Construction of mono-dimensional utility function

Let us describe in this section the usual case where the preferences of the DM over viewpoint k depends only on attribute X_k. We denote by $\succeq_k$ the preference relation of the DM over X_k. Under the Birkhoff-Milgram theorem, if $\succeq_k$ is a weak order and $X_k/\sim_k$ contains a countable order-dense subset, then $\succeq_k$ can be represented by a function $u_k : X_k \to \mathbb{R}$, that is [7]

$$x \succeq_k x' \iff u_k(x) \geq u_k(x') \; . \qquad (1)$$

Function u_k satisfying (1) corresponds to an ordinal scale. In order to obtain a cardinal scale, one needs to possess a quaternary relation $\succeq_k^\star$ over $X_k \times X_k$. For $x, x', t, t' \in X_k$, $x\,x' \succeq_k^\star t\,t'$ means that the intensity of preference between x and x' is at least as good as that between t and t'. Under some reasonable conditions on $\succeq_k^\star$, we have the following equivalence [7]

$$xx' \succeq_k^\star tt' \iff u_k(x) - u_k(x') \geq u_k(t) - u_k(t') \; . \quad (2)$$

AHP [9] and Macbeth [3] can be used to construct utility functions. We only describe Macbeth since it is well-founded from a measurement standpoint, and ensures both (1) and (2). Assume that X_k is an interval of $\mathbb{R}$. Utility function u_k is assumed to be piecewise linear. It is thus enough to determine u_k on a finite set of elements $\{x^1, \ldots, x^q\} \subset X_k$, with $x^1 \leq \cdots \leq x^q$. u_k is indeed affine between two points x^i and x^{i+1}:

$$u_k(x) = I(x; x^1, u_k(x^1), \ldots, x^q, u_k(x^q))$$

where

$$I(x; x^1, u^1, \ldots, x^q, u^q) = \begin{cases} u^1 & \text{if } x \leq x^1 \\ u^i + (u^{i+1} - u^i)\,\frac{x - x^i}{x^{i+1} - x^i} & \text{if } x^i \leq x < x^{i+1} \\ u^q & \text{if } x \geq x^q \end{cases}$$

There remains to explain how $u_k(x^1), \ldots, u_k(x^q)$ are determined in the Macbeth approach. The construction of $\succeq_k^\star$ on the q selected elements $x^1, \ldots, x^q$ requires the comparison of $q(q-1)/2$ quantities (namely all differences $u_k(x^j) - u_k(x^l)$ where $x^j \succ_k x^l$), which demands roughly $q^4/8$ operations. In order to reduce the complexity, the idea is ask the DM to assess directly the $q(q-1)/2$ differences of satisfaction $u_k(x^j) - u_k(x^l)$ between two values x^j and x^l, for any $j \neq l$ such that $x^j \succ_k x^l$. Since the DM cannot assess such quantities with precise numbers, the Macbeth approach proposes an ordinal assessment scales composed of only 6 elements: {*very small, small, mean, large, very large, extreme*}=: $\mathcal{E}$. The information asked in practice is thus quite similar to $\succeq_k^\star$. The advantage of asking $u_k(x^j)-u_k(x^l)$ is that it leads to a redundant information that will enable to compute accurate but non-unique values of $u_k(x^j)$ for $j \in \{1,\ldots,q\}$. The second advantage is that it is easier for a human being to give some relative information regarding a difference (for instance $u_k(x^j) - u_k(x^l)$) than to give some absolute information (for instance $u_k(a^j)$).

Previous method defines an interval scale. Such scale is unique up to an affine transformation. u_k can be uniquely determined by fixing its value at two reference points of X_k. We assume that there exists on X_k an element denoted by $\mathbf{0}_k$ that is considered as completely unacceptable by the DM, and an element denoted by $\mathbf{1}_k$ that is considered perfectly satisfactory by the DM. In order to specify entirely utility function u_k, the utilities of reference elements $\mathbf{0}_k$ and $\mathbf{1}_k$ are fixed to values 0 and 1 respectively.

$$u_k(\mathbf{0}_k) = 0 \quad , \quad u_k(\mathbf{1}_k) = 0 \ . \tag{3}$$

3 Naive ways to construct multi-dimensional utility functions

The first naive method to construct multi-dimensional utility function $u_k : X_k \times Y_k^1 \times \cdots \times Y_k^{p_k} \to \mathbb{R}$ consists in applying exactly the approach described in Section 2. Utility function u_k can indeed be seen as a function on the Cartesian product $X_k \times Y_k^1 \times \cdots \times Y_k^{p_k}$. Assume that each set $X_k, Y_k^1, \ldots, Y_k^{p_k}$ can be characterized by q elements. Hence one needs to determine u_k on q^{p_k+1} elements to determine u_k everywhere. Applying Macbeth, this means that the DM must provide $\frac{q^{p_k+1}(q^{p_k+1}-1)}{2} \approx \frac{1}{2} q^{2(p_k+1)}$. The generalization of mono-dimensional approaches to the multi-dimensional case would lead to a number of questions that is far too big for the DM. So this method is not possible in practice.

An alternative solution would be to synthesize all variables into one unique. In previous example, let us denote by x the sighting shift, by y_1 the distance to target and by y_2 the speed of the target. This amounts to defining a new variable such as

$$z = x - \frac{1}{10}\, y_1 - \frac{1}{3}\, y_2$$

and construct a one-dimensional utility function on z. This is what B. Roy have called *arithmo-morphism* [8]. The construction of z is very empiric and arbitrary [8].

We propose in this paper a method to construct a utility function $u_k : X_k \times Y_k^1 \times \cdots \times Y_k^{p_k} \to \mathbb{R}$ that is not doing arithmo-morphism, with a number of data provided by the DM which is not much more than what is done for usual mono-dimensional utility functions.

4 Representation of the multi-dimensional utility function

Point of view $k \in N$ is described by a main variable x called metric which values belong to the set X_k. The interpretation of each value of that metric depends on a context depicted by a set $y_1, \ldots, y_{p_k}$ of contextual variables taking values in the sets $Y_k^1, \ldots, Y_k^{p_k}$ respectively.

We denote by $\succeq_k$ the preference relation of the DM over $X_k \times Y_k^1 \times \cdots \times Y_k^{p_k}$. Set $y = (y_1, \ldots, y_{p_k}) \in Y_k := Y_k^1 \times \cdots \times Y_k^{p_k}$. In the following, for $S \subseteq P_k$, we will denote (y_S, y'_{-S}) the compound vector z with $z_j = y_j$ for $j \in S$ and $z_j = y'_j$ for $j \in P_k \setminus S$. For $y \in Y_k$ fixed, define the preference relation $\succeq_{k,y}$ over X_k, which is just the restriction of $\succeq_k$ on the set $X_k \times \{y\}$.

Preference relation $\succeq_{k,y}$ is a mono-dimensional order relation that can be constructed as described in Section 2. By (1), (2) and (3), we obtain

$$x \ \succeq_{k,y} \ x' \iff U_{k,y}(x) \geq U_{k,y}(x') \ . \tag{4}$$

$$x\, x' \ \succeq_{k,y}^\star \ t\, t' \iff U_{k,y}(x) - U_{k,y}(x') \geq U_{k,y}(t) - U_{k,y}(t') \ . \tag{5}$$

$$\exists \mathbf{0}_{k,y}, \mathbf{1}_{k,y} \in X_k \ , \quad U_{k,y}(\mathbf{0}_{k,y}) = 0 \quad , \quad U_{k,y}(\mathbf{1}_{k,y}) = 0 \ . \tag{6}$$

Recall that $\mathbf{0}_{k,y}$ and $\mathbf{1}_{k,y}$ correspond to two reference points.

The preferences of the DM on viewpoint k is mainly influenced by attribute X_k. The contextual variables just strengthen or weaken this link. They cause the judgment of the DM being more or less tolerant. One feels that the preferences as described by $\succeq_{k,y}$ can be derived from $\succeq_{k,y'}$. More precisely, we look for an homomorphism from $(X_k, \succeq_{k,y})$ and $(X_k, \succeq_{k,y'})$. Under some order-dense conditions between $(X_k, \succeq_{k,y})$ and $(X_k, \succeq_{k,y'})$, which states that $(X_k, \succeq_{k,y})$ and $(X_k, \succeq_{k,y'})$ are of the same dimension, there exists an homorphism $\phi_{y,y'} : X_k \to X_k$ such that

$$\forall x, x' \in X_k \ , \quad x \succeq_{k,y} x' \iff \phi_{y,y'}(x) \succeq_{k,y'} \phi_{y,y'}(x') \ . \tag{7}$$

$\phi_{y,y'}(x)$ is the value of X_k at context y' that is equivalent to value x at context y.

$$\forall y, y' \in Y_k \quad , \quad (x, y) \sim_k (\phi_{y,y'}(x), y') \ , \tag{8}$$

Function $\phi_{y,y'}$ shall be surjective so that $Range(\phi_{y,y'}) = X_k$. Function $\phi_{y,y'}$ shall be strictly increasing so that $\succeq_{k,y}$ and $\succeq_{k,y'}$ represent the same ordinal preferences up to shrinking and dilations of the X_k scale.

Since $\phi_{y,y'}$ is an homomorphism, it shall preserve the reference elements :

$$\forall x \in X_k \quad \phi_{y,y'}(\mathbf{0}_{k,y'}) = \mathbf{0}_{k,y} \quad , \quad \phi_{y,y'}(\mathbf{1}_{k,y'}) = \mathbf{1}_{k,y} \; . \tag{9}$$

Lemma 1 *Under (4), (5), (6), (8) and (9), one has for all $x \in X_k$ and $y, y' \in Y_k$*

$$U_{k,y}(x) = U_{k,y'}(\phi_{y,y'}(x)) \; .$$

Previous lemma shows that the utility function for one context y' can be deduced from the utility function for another context y.

We now assume that $\phi_{y,y'}$ is continuous and strictly increasing, and that X_k is an interval (X_k is convex in $\mathbb{R}$).

Lemma 2 *For all $x \in X_k$ where $L(x) = \{x' \in X_k \; , \; U_{k,y}(x') = U_{k,y}(x)\}$ is finite, then*

$$\phi_{y,y'}(x) = {\phi_{y',y}}^{-1}(x) \; .$$

For the other values of x, $U_{k,y}$ is constant and if $\phi_{y,y'}(x) = {\phi_{y',y}}^{-1}(x)$, one would have exactly the same utility value.

One can assume that for all $x \in X_k$

$$\phi_{y,y'}(x) = {\phi_{y',y}}^{-1}(x) \; . \tag{10}$$

Previous lemma shows that it is not necessary to construct the interval scale $U_{k,y}$ for all $y \in Y_k$. If transformation ϕ is known, one needs only to construct the interval scale for only one $y \in Y_k$. This particular element $y^\star$ of Y_k is called the *reference context*. This reference context corresponds to a usual situation where the DM can easily construct the interval scale $U_{k,y^\star}$. We construct the utility function $U_{k,y^\star}$ using what is described in Section 2 since it corresponds to a mono-dimensional utility function. One needs to identify in X_k several elements $\{x^1, \ldots, x^q\} \subset X_k$, with $x^1 \leq \cdots \leq x^q$. From, what is described in Section 2, the value of $U_{k,y^\star}$ is determined at those elements. Then we obtain

$$U_{k,y^\star}(x) = I(x; x^1, U_{k,y^\star}(x^1), \ldots, x^q, U_{k,y^\star}(x^q))$$

If transformation $\phi_{y,y^\star}$ is known, then one obtains $U_{k,y}$ for any $y \in Y_k$ from the following relation

$$U_{k,y}(x) = U_{k,y^\star}(\phi_{y,y^\star}(x)) \; . \tag{11}$$

From (10) and Lemma 1, one obtains the following result.

Lemma 3 *We have*

$$U_{k,y}(x) = I(x; \phi_{y^\star,y}(x^1), U_{k,y^\star}(x^1), \ldots, \phi_{y^\star,y}(x^q), U_{k,y^\star}(x^q)) \tag{12}$$

It proves that one needs only to specify function $y \mapsto \phi_{y^\star,y}(x^i)$ for all $i \in \{1, \ldots, q\}$, which is the aim of Section 5.

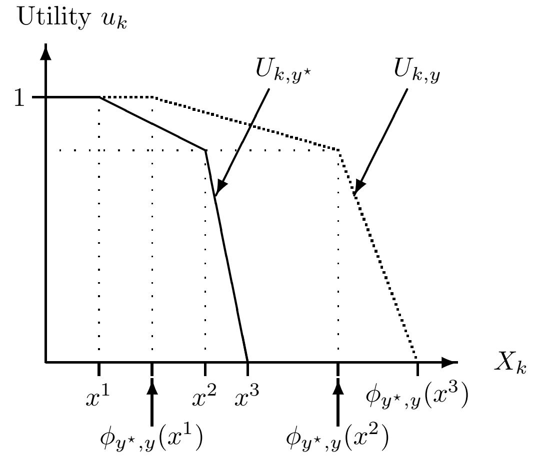

Figure 1: Fuzzy levels $N_{-m}, \ldots, N_0, \ldots, N_m$.

5 Interpolation

5.1 Approach

From Lemma 3, there remains to define $y \mapsto \phi_{y^\star,y}(x^i)$ for all $i \in \{1, \ldots, q\}$. Let us fix one $i \in \{1, \ldots, q\}$ and one $y \in Y_k$.

Function $y \mapsto \phi_{y^\star,y}(x^i)$ can be assumed to be piecewise linear. More precisely, one can assume that $\phi_{y^\star,y}(x^i)$ is known at some values $y(1), \ldots, y(r) \in Y_k$ of y, and $\phi_{y^\star,y}(x^i)$, for any y is computed by interpolation.

One can simply suppose that the points $y(1), \ldots, y(r)$ define a regular rectangular mesh in Y_k. The main advantage of this representation is that the interpolation in a rectangular mesh is simple to compute and intuitive for the DM. However, even if there are only a few elements on each contextual variable, this yields a huge number of points in the mesh. This leads to a huge number of information that the DM has to provide.

The idea is thus to consider any set $y(1),\dots,y(r)$ of points in Y_k. We suppose that the DM has given the values of $y(1),\dots,y(r)$ as well as the associated values $\phi_{y^\star,y(1)}(x^i) =: x^i(1)$, …, $\phi_{y^\star,y(r)}(x^i) =: x^i(r)$. Hence the DM has to provide the following array.

Selected values on X_k	Contextual variables Y_k		
$x^i(1)$	$y_1(1)$	$\cdots$	$y_{p_k}(1)$
$x^i(2)$	$y_1(2)$	$\cdots$	$y_{p_k}(2)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x^i(r)$	$y_1(r)$	$\cdots$	$y_{p_k}(r)$

Computing $\phi_{y^\star,y}(x^i)$ for any y from previous array is an interpolation problem. Our concern is about the choice of the interpolation. If one chooses to use a multi-dimensional interpolation on Y_k, the result becomes very hard to predict, and thus the DM will not master the consequences of the data he provides.

We define thus an interpolation that is constructed iteratively so that the DM has a better grasp on the result. This interpolation is based on an ordering of the contextual variables from the most important ones to the less important. Let τ be a permutation on P_k.

Selected values on X_k	Contextual variables Y_k		
$x^i(1)$	$y_{\tau(1)}(1)$	$\cdots$	$y_{\tau(p_k)}(1)$
$x^i(2)$	$y_{\tau(1)}(2)$	$\cdots$	$y_{\tau(p_k)}(2)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x^i(r)$	$y_{\tau(1)}(r)$	$\cdots$	$y_{\tau(p_k)}(r)$

For $t \in \{1,\dots,p_k\}$, we define the *Lexicographic* ordering $\preceq^t_{\text{lexico}}$ by

$$y \preceq^t_{\text{lexico}} y' \;\Leftrightarrow\; \exists s \le t \text{ such that } \forall l \in \{1,\dots,s\}\; y_{\tau(l)} = y'_{\tau(l)} \text{ and } y_{\tau(s)} < y'_{\tau(s)}$$

for any $y, y' \in Y_k$. Let us denote by $\prec^t_{\text{lexico}}$ be the asymmetric part of $\preceq^t_{\text{lexico}}$, and by $\sim^t_{\text{lexico}}$ the symmetric part of $\preceq^t_{\text{lexico}}$. One assumes that $y(1),\dots,y(r)$ are ordered according to the lexicographic ordering

$$y(1) \prec^{p_k}_{\text{lexico}} y(2) \prec^{p_k}_{\text{lexico}} \cdots \prec^{p_k}_{\text{lexico}} y(r)\ .$$

The interpolation is performed one variable at a time, starting with variable $\tau(p_k)$, then $\tau(p_k-1)$, …, up to $\tau(1)$. Let us describe the process for variable $\tau(p_k)$. We begin by grouping the elements $y(1),\dots,y(r)$ into clusters in which all elements have the same value on variables $y_{\tau(1)},\dots,y_{\tau(p_k-1)}$ within each cluster. More precisely, let us define $h_0=1,h_1,\dots,h_m$ such that

$$\begin{aligned} & y(1) \sim^{p_k-1}_{\text{lexico}} y(2) \sim^{p_k-1}_{\text{lexico}} \cdots \sim^{p_k-1}_{\text{lexico}} y(h_1-1) \\ & \prec^{p_k-1}_{\text{lexico}} y(h_1) \sim^{p_k-1}_{\text{lexico}} \cdots \sim^{p_k-1}_{\text{lexico}} y(h_2-1) \\ & \prec^{p_k-1}_{\text{lexico}} y(h_2)\ \cdots\ y(h_{m-1}-1) \\ & \prec^{p_k-1}_{\text{lexico}} y(h_{m-1}) \sim^{p_k-1}_{\text{lexico}} \cdots \sim^{p_k-1}_{\text{lexico}} y(h_m-1)\ . \end{aligned}$$

The m clusters are thus $\{y(1),\dots,y(h_1-1)\}$, $\{y(h_1),\dots,y(h_2-1)\}$, …, $\{y(h_{m-1}),\dots,y(h_m-1)\}$. From value $y_{\tau(p_k)}$ of the last variable in the order τ, we can transform each cluster into one single element of $Y_k \setminus Y_k^{p_k}$. More precisely, one can transform previous array in the following one, removing last variable $\tau(p_k)$

values on X_k	Contextual variables Y_k		
$\tilde{x}^i(1)$	$y_{\tau(1)}(h_0)$	$\cdots$	$y_{\tau(p_k-1)}(h_0)$
$\tilde{x}^i(2)$	$y_{\tau(1)}(h_1)$	$\cdots$	$y_{\tau(p_k-1)}(h_1)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\tilde{x}^i(m)$	$y_{\tau(1)}(h_{m-1})$	$\cdots$	$y_{\tau(p_k-1)}(h_{m-1})$

where for all $l \in \{1,\dots,m\}$

$$\begin{aligned} \tilde{x}^i(l) = I(& y_{\tau(p_k)}; y_{\tau(p_k)}(h_{l-1}), x^i(h_{l-1}), \\ & y_{\tau(p_k)}(h_{l-1}+1), x^i(h_{l-1}+1), \dots, \\ & y_{\tau(p_k)}(h_l-1), x^i(h_l-1)) \end{aligned}$$

The same process is applied to previous array in order to remove the last remaining variable in the order τ, i.e. variable $\tau(p_k-1)$. And so on. When there remains only variable $\tau(1)$

values on X_k	Contextual variables Y_k
$\hat{x}^i(1)$	$\hat{y}_{\tau(1)}(1)$
$\hat{x}^i(2)$	$\hat{y}_{\tau(1)}(2)$
$\cdots$	$\cdots$
$\hat{x}^i(w)$	$\hat{y}_{\tau(1)}(w)$

then we obtain

$$\begin{aligned} \phi_{y^\star,y}(x^i) = I(& y_{\tau(1)}; \hat{y}_{\tau(1)}(1), \hat{x}^i(1), \hat{y}_{\tau(1)}(2), \hat{x}^i(2), \\ & \dots, \hat{y}_{\tau(1)}(w), \hat{x}^i(w)) \end{aligned}$$

5.2 Example

To illustrate our approach, consider once more the example of shooting exercise on a simulator. Y_1 corresponds to the speed and Y_2 corresponds to the distance. The reference context is defined at fixed target at short range, i.e. $y_1^\star = 0\,m/s$ and $y_2^\star = 10\,m$.

The application of MACBETH leads to the following values of x^i and $U_{k,y^\star}(x^i)$.

Values of the utilities			
1	0.7	0.2	0
Selected values on X_k			
$10\,cm$	$15\,cm$	$20\,cm$	$30\,cm$

One notices that the smaller the value of the attribute the better.

Speed is more important than Distance. The DM provides the following values that specify $\phi_{y^\star,y}$.

Selected values on X_k				Speed	Distance
10 cm	15 cm	20 cm	30 cm	0 m/s	10 m
15 cm	20 cm	25 cm	40 cm	10 m/s	10 m
20 cm	25 cm	30 cm	45 cm	20 m/s	10 m
30 cm	35 cm	40 cm	55 cm	20 m/s	30 m
20 cm	25 cm	30 cm	45 cm	20 m/s	50 m

Note that the first line corresponds to the reference. The contextual variables are ordered in such a way that Speed comes first and Distance comes second by τ.

When shooting at a static target, the judgment does not depend on the distance. Even if aiming is more tedious for long range than short range, the instructor does not want to take this into account in his evaluation. This applies also for medium speed. However, in this case, the expert is a little more tolerant. One indeed notice that the values of the selected values are increased. Now at large speed (20 m/s), the judgment now depends on the distance. At low and long range, the DM is more tolerant than for medium range. The reason is that the "defilement" speed is large at small range, which makes aiming difficult; and aiming at long range is always difficult, especially when the target is moving.

Let us explain now how the utility u_k is computed for values $x = 28\ cm$, $y_1 = 15\ m/s$ and $y_2 = 20\ m$. The removal of last contextual variable leads to the following array.

Selected values on X_k				Speed
10 cm	15 cm	20 cm	30 cm	0 m/s
15 cm	20 cm	25 cm	40 cm	10 m/s
25 cm	30 cm	35 cm	50 cm	20 m/s

As example, the last selected value for the reduced context $y_1 = 20\ m/s$ is $I(20m; 10m, 45cm; 30m, 55cm; 100m, 60cm) = 50cm$.

Repeating this on the last contextual variable, we obtain $\phi_{y^\star,y}$

Selected values on X_k			
20 cm	25 cm	30 cm	45 cm

We conclude that the utility is $I(28cm; 20cm, 1, 25cm, 0.7, 30cm, 0.2, 45cm, 0) = 0.4$.

6 Explanation of the utility function on real data

We have seen that the computation of $u_k(x, y_1, \ldots, y_{p_k}) =: v$ results from several nested interpolations. Value v taken by criterion k is not trivial to the DM. The DM is thus interested in an explanation of the reason why u_k equals v. It is completely useless to explain the precise value of v. Yet, the user wants to know why v is, for instance, rather bad, rather mean or rather good. This amounts thus to explaining why v belongs to a fuzzy set. Suppose thus that the $[0, 1]$ satisfaction scale is covered by $2\,m + 1$ fuzzy levels $N_{-m}, \ldots, N_0, \ldots, N_m$ described by the fuzzy membership functions $\mu_{N_{-m}}, \ldots, \mu_{N_m}$ (See Figure 2). Level $\mu_{N_{-m}}$ is the worse level (e.g. *very bad*), μ_{N_m} is the best level (e.g. *very good*).

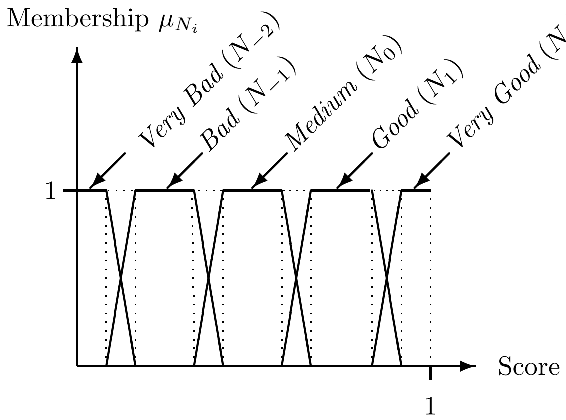

Figure 2: Fuzzy levels $N_{-m}, \ldots, N_0, \ldots, N_m$.

We begin by determining the level corresponding to value v. This is $i = \mathrm{Argmax}_{j\in\{-m,\ldots,m\}} \mu_{N_j}(v)$.

6.1 Main argument

The main argument concerns the relation between x and u_k at fixed value of the context $y = (y_1, \ldots, y_{p_k})$. It is not possible to provide a true argumentation of the evaluation made. The explanation we propose to give consists in the interval of values of attribute k around x for which the assessment would belong to the same fuzzy level i for the same context y.

We first determine the largest interval of values $(\underline{v}, \overline{v})$ around v such that for all $t \in (\underline{v}, \overline{v})$, $\mu_{N_i}(t) = \max_{j\in\{-m,\ldots,m\}} \mu_{N_j}(t)$. Then, we determine the largest interval $(\underline{x}, \overline{x})$ such that

$$x \in (\underline{x}, \overline{x})$$

and

$$\forall x' \in (\underline{x}, \overline{x})\ ,\ u_k(x', y_1, \ldots, y_{p_k}) \in (\underline{v}, \overline{v})$$

The text that is displayed to the user is thus

Between $\underline{x}$ and $\overline{x}$, the value of attribute k is judged N_i.

Note that values $\underline{x}$ and $\overline{x}$ used in the explanation do not correspond to the values that were given by the expert in the specification of the utility function.

6.2 Argument on the secondary variables

The second part of the explanation concerns the influence of the contextual variables on the evaluation. The idea is to compare the result $v := u_k(x, y)$ to what would have been obtained $v^\star = u_k(x, y^\star)$ with the reference context $y^\star$. If v is significantly larger than $v^\star$, the judgment is more tolerant in the current context y than what would have been obtained in the reference context $y^\star$. If v is significantly smaller than $v^\star$, the judgment is more intolerant in the current context than what would have been obtained in the reference context. We quantify the difference $v - v^\star$, among $2m+1$ fuzzy levels $Q_{-m}, \ldots, Q_0, \ldots, Q_m$ that cover interval $[-1, 1]$. Let μ_{Q_i} be the membership of Q_i. One can set $\mu_{Q_i}(z) = \mu_{N_i}((z+1)/2)$ for all $z \in [-1, 1]$. We begin by determining the level corresponding to value $v - v^\star$. This is the level $g = \text{Argmax}_{j \in \{-m,\ldots,m\}} \mu_{Q_j}(v - v^\star)$.

We have the following three cases:

- If $g = 0$, then v is not significantly different from $v^\star$. Hence the context has not played an important role in the evaluation.
- If $g > 0$, the judgment is more tolerant in the current context than what would have been obtained in the reference context. Like for the main argument, the idea is to give to the user the values of the contextual variables for which the difference $v - v^\star$ would also belong to level Q_g. However, doing so means that we show to the user why the difference belongs to Q_g and not to Q_{g-1}, and why the difference belongs to Q_g and not to Q_{g+1}. The second argument is not relevant to the case $g > 0$. Hence we need to explain why the difference belongs to Q_g and not to Q_{g-1}, that is why the difference belongs to Q_g or Q_{g+1}, …, or Q_m. More precisely, one is interested in the values y'_j of variable j for which $u_k(x, y-_j, y_{-j}) - v^\star$ belongs to level Q_g or Q_{g+1}, …, or Q_m. We first determine the smallest values $\underline{U_g}$ such that for all $t \geq \underline{U_g}$, $\text{Argmax}_{l \in \{-m,\ldots,m\}} \mu_{Q_l}(t) \geq g$. One looks thus for the largest interval $(\underline{y_j}, \overline{y_j})$ such that $y_j \in (\underline{y_j}, \overline{y_j})$ and
$$\forall y'_j \in (\underline{y_j}, \overline{y_j}) \ , \ u_k(x, y_j, y_{-j}) - v^\star \geq \underline{U_g}$$
- If $g < 0$, the judgment is more intolerant in the current context than what would have been obtained in the reference context. Like previous case, the idea is to give to the user the values of the contextual variables for which the difference would also belong at most to level Q_g, i.e. to levels Q_g, or $Q_{g-1}, \ldots, Q_1$. More precisely, one is interested in the values y'_j of variable j for which $u_k(x, y-_j, y_{-j}) - v^\star$ belongs to level Q_g or Q_{g-1}, …, or Q_{-m}. We first determine the smallest values $\overline{U_g}$ such that for all $t \leq \underline{U_g}$, $\text{Argmax}_{l \in \{-m,\ldots,m\}} \mu_{Q_l}(t) \leq g$. One looks thus for the largest interval $(\underline{y_j}, \overline{y_j})$ such that $y_j \in (\underline{y_j}, \overline{y_j})$ and
$$\forall y'_j \in (\underline{y_j}, \overline{y_j}) \ , \ u_k(x, y-_j, y_{-j}) - v^\star \leq \overline{U_g}$$

All variables do not have the same influence on the utility. Thus, it is not relevant to display to the user the intervals $(\underline{y_j}, \overline{y_j})$ for all variables j. When $g \neq 0$, it makes sense to present to the DM the intervals $(\underline{y_j}, \overline{y_j})$ only for the variables $j \in P_k$ that have a significant importance in the difference $v - v^\star$.

We wish thus to assess the contribution of each variable to the difference $v - v^\star$. It is not easy to differentiate the real contribution of one variable irrespectively of the other ones since all variables vary. Let π be a permutation on P_k. Variables of P_k are considered one at a time in the order given by π. We start with the situation where the contextual variables are set to the reference values $y^\star$. Then each time a variable is considered, the value of this variable is turned to the current value y, and the difference that is obtained is defined as the contribution of this variable. For $S \subseteq P_k$, set $V(S) = u_k(x, y_S, y^\star_{-S})$. The first variable is $\pi(1)$. Its contribution is
$$C_\pi(\pi(1)) := V(\{\pi(1)\}) - V(\emptyset) \ .$$
Likewise, the contribution of the l^{th} variable is
$$\begin{aligned} C_\pi(\pi(l)) =& V(\{\pi(1), \ldots, \pi(l-1), \pi(l)\}) \\ & - V(\{\pi(1), \ldots, \pi(l-1)\}) \end{aligned}$$
One has
$$\sum_{l \in P_k} C_\pi(\pi(l)) = V(N) - V(\emptyset) = v - v^\star \ .$$
Since there is no a priori reason to chose one particular permutation π, the contribution of one variable is defined as the contribution over all possible permutations
$$C(j) = \frac{1}{2^{p_k}} \sum_\pi C_\pi(j) \ .$$
One gets
$$C(j) = \sum_{S \subseteq P_k} \frac{|S|!(p_k - |S| - 1)!}{(p_k)!} \ (V(S \cup \{j\}) - V(S))$$
This expression of $C(j)$ corresponds to the Shapley value of capacity V for variable j [10]. One has
$$\sum_{j \in P_k} C(j) = V(N) - V(\emptyset) = v - v^\star \ .$$

Hence the mean contribution of a variable is $(v - v^\star)/p_k$. The variables that have a valuable contribution to the difference $v - v^\star$ are thus the variables $j \in P_k$ such that $C(j) \geq (v - v^\star)/p_k$.

To sum-up, when $g \neq 0$, we generate the following sentence:

> `The judgment is` Q_g`-more tolerant (if` $g > 0$`)/`Q_g`-more intolerant (if` $g < 0$`) for a value on variable` j `between` $\underline{y_j}$ `and` $\overline{y_j}$ `(if` $C(j) \geq (v - v^\star)/p_k$`).`

6.3 Example

To illustrate our approach, we go back to the example given in Section 5.2.

Consider values $x = 28\ cm$, $y_1 = 15\ m/s$ and $y_2 = 20\ m$. One has $u_k = 0.4$ and the explanation that is generated is:

```
Between 26cm and 28cm, 'sighting
shift' is judged 'mean' in the
context "Target Speed=15 m/s" and
"Distance=20 m". The judgment is
more tolerant for a value of 'Target
Speed' greater than 13m/s.
```

Consider values $x = 15cm$, $y_1 = 15m/s$ and $y_2 = 20m$. One has $u_k = 1$ and the explanation that is generated is:

```
Under 24cm, 'sighting shift'
is judged 'very good' in the
context "Target Speed=15 m/s" and
"Distance=20 m". The judgment is
more tolerant for a value of 'Target
Speed' greater than 6m/s.
```

Consider values $x = 26\ cm$, $y_1 = 6\ m/s$ and $y_2 = 45\ m$. One has $u_k = 0.17$ and the explanation that is generated is:

```
Above 23cm, 'sighting shift'
is judged 'very bad' in the
context "Target Speed=6 m/s" and
"Distance=45 m". The judgment does
not depend on 'Distance'.
```

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