Two consensus protocols based on an acceptance threshold in Group Decision Making

Christophe Labreuche
THALES Research and Technology
RD 128, 91767 Palaiseau Cedex, France
christophe.labreuche@thalesgroup.com

Abstract

We define two negotiation protocols for Group Decision Making, which main feature is the existence of acceptation thresholds. In order to predict which consensus are expected to arise, we study the equilibria of these protocols in the sense of Game Theory. We also investigate their sensitivity to manipulation by the actors.

Keywords: Group Decision Making, Preferences, Game Theory, Equilibria, Manipulation.

1 Introduction

By Group Decision Making (GDM in short) we mean a set of decision makers (DM) that have jointly to single out one option among several from the preferences of each DM. These preferences are assumed to be represented by a cardinal utility function.

A first class of approaches to previous problem consists in defining an automatic procedure, also called mechanism, in which the DMs do not participate directly. The most well-known ones are derived from voting rules, using Fuzzy Set Theory [10]. A GDM problem is indeed similar to voting problems, except that the preference representation is not cardinal in these latters [1]. One can also find extensions of some concepts of Cooperative Game Theory, such as the Core or the bargaining set, to GDM problems [17]. Whatever the solution that is proposed, there is always the possibility that a DM complains about the outcome of the mechanism. This leads to the second class of approaches where the idea is to let the DMs discuss together and reach by themselves a consensus. The GDM community has mainly focuses on defining measures of soft consensus [11], on understanding the reason why individual preferences differ and on providing recommendations to the DMs who are the most distant to the consensual position, of how to change their preferences to reduce the overall disagreement [16]. Another reason why the preferences of the individuals can vary over time is the fact that each agent is influenced by the other ones. From a theoretical standpoint, one can assume that such influence is known and modeled with, for instance De Groot's linear model [3] or the soft consensus non linear model [6]. Consensual dynamics consists then in predicting what will be the result of such influences.

In the Multi-Agent and Artificial Intelligence communities, the problem of finding a consensus is seen as a bargaining problem where the consensus results from a negotiation *protocol* [14, 9, 5]. This approach has not received much attention in the GDM community [4]. The goal of this paper is to study a type of protocol that tackle the GDM problem.

A consensus protocol is an iterative process that formalizes the exchanges that are allowed among the DMs. At each run of a protocol, one proposal is made by one DM or a group of DMs, the other DMs vote on the acceptation or rejection of that offer, and the process goes on if the offer is rejected.

We consider two protocols in this paper. In these two protocols, the main feature is the existence of an acceptance threshold such that any offer more attractive for a DM than that threshold must be accepted by him. This threshold decreases over time, compelling the DMs to accept more and more offers over time. The only difference between these two protocols is that individual DMs are actors of the protocol in the first one whereas coalitions of DMs are actors in the second protocol.

It is not easy to see whether a protocol is a priori expected to lead to fair consensual options. Non-Cooperative Game Theory is a powerful tool to perform such prediction under rationality assumptions. One indeed looks for the *equilibria* of the protocols. Interestingly, the equilibria for the first protocol lead to the usual maximin solution, whereas the second protocol leads to a weak version of the counterpart in GDM problems of the Nucleolus known in Cooperative Game Theory [13].

Another important issue often studied for voting rules is the possibility for the DMs to manipulate the outcome of the consensus by announcing preferences that are not their true ones [7, 15, 12]. Our two protocols are easily manipulable. The origin comes from the cardinal nature of the preference representation. Two possible proposals to reduce this sensitivity are made.

The rest of the paper is organized as follows. Section 2 gives some preliminaries. The first protocol is described and analyzed in Section 3. The sensitivity of this protocol to manipulations is discussed in Section 4. Section 5 depicts the second protocol.

2 Description of the problem

Let N be the set of DMs and A the set of options. A is supposed to be finite. We assume that the preferences of DM i can be expressed as a utility function u_i such that the larger the better for DM i. Without loss of generality, utilities can be assumed to belong to the $[0, 1]$ interval. The set of functions from A onto $[0, 1]$ is denoted by $\mathcal{U}(A)$. A consensus mechanism is a function $M : \mathcal{U}(A)^n \rightarrow A$.

It is not easy to predict the behavior of the DMs face to a negotiation protocol. *Subgame Perfect Equilibria* (SPE) [13] give realistic predictions under the assumption that DMs behave rationally and that they have perfect information on all preferences. The difficulty comes from the fact that, during the protocol, a DM cannot readily assess the consequences of his choices since he does not know the choice of the next DMs in the protocol. The only case where this is simple is for the last DM to give a decision. The second last DM can then assess the consequences of his choices, assuming that the last DM considers his best choices. And so on. This backward induction algorithm defines SPEs [13].

3 Description and analysis of the first protocol

In a negotiation protocol, a DM may stick to his initial preferences without taking into account the arguments of the other players. As a result, the protocol never converges to an agreement. In the literature, it is often assumed that options become less and less attractive over time so that early consensus are encouraged [14]. However, assuming that the utility of DMs for an option decays with the time is not always valid or may be complex to represent in practice. Another possibility is to force DMs to accept proposals for which their utility is larger than a given threshold. This threshold is large at the beginning of the protocol so that players have the complete freedom to refuse, and the threshold decreases with time, enforcing players to accept more and more potential proposals, and thus to make more and more concession. This reduction of the freedom of the DMs is motivated by fairness (it is not fair that a player accepts a not so good option to him and that another player rejects an option with a similar utility) and cooperation (make sure to converge to an agreement) reasons.

We assume that the order in which DMs are proposers is defined beforehand. More precisely, DM τ_k is the proposer at iteration k. We introduce the following protocol.

Protocol ***P1***. At iteration $k \in \mathbb{N}_*$:

- Player τ_k makes an offer $a \in A$.
- All other players say *Yes* or *No*. However, a player $i \in N \setminus \{\tau_k\}$ must necessarily accept if $u_i(a) \geq \rho_i(k)$.
- If at least one player rejects the offer, we go to the next iteration. Otherwise, a is chosen and the protocol ends.

The option that is chosen is called *outcome* of the protocol. We introduce the following condition.

Axiom **Thresholds (T)**. For all $i \in N$, $\rho_i(k)$ is non-increasing w.r.t. k and reaches value 0 in a finite time $T^\star$ (i.e. $\rho_i(T^\star) = 0$).

Thanks to that axiom, after a given time (that is large enough), any proposal is necessarily accepted by all players, so that ***P1*** ends after a finite time.

Lemma 1 *Under* **(T)***, protocol* ***P1*** *has at least one SPE.*

Let $A^\star(k) = \{a \in A\ ,\ \forall i \in N\ \ u_i(a) \geq \rho(k)\}$. At an iteration k, any offer in $A^\star(k)$ is necessarily accepted as the outcome. By **(T)**

$$A^\star(1) \subseteq A^\star(2) \subseteq \cdots \subseteq A^\star(T^\star) = A\ .$$

Lemma 2 *Assume that* **(T)** *holds. Let K be the smallest integer such that $A^\star(K) \neq \emptyset$. Assume furthermore that all players are proposer at least once at or after iteration K. Let K' be the smallest integer larger than K such that*

$$\{\tau_K, \tau_{K+1}, \ldots, \tau_{K'}\} = N\ .$$

Then the resulting outcome of any SPE of protocol ***P1*** *belongs to $A^\star(K')$.*

Player τ_K that is proposer at iteration K is the first player that can be sure to get an acceptable offer. However the options of $A^\star(K)$ may not be very attractive to him so that he can make an unacceptable offer and hope that a better option will be proposed at a further iteration.

The following result shows that we can restrict ourself to SPEs that ends at iteration K'. The other SPEs use unnecessary extra iterations.

Corollary 3 *Under the assumptions of Lemma 2, for any SPE of protocol **P1**, there exists a SPE of protocol **P1** that ends at an iteration in $\{1, \dots, K'\}$ and that leads to the same outcome.*

By Corollary 3, it is enough for restrict to iteration in $\{K, \dots, K'\}$, in the computation of the SPE. We define the following algorithm.

Algorithm ***Algo4***. Let us define $R_k \subseteq A$ on induction of $k \in \{1, \dots, K'\}$ as follows.

- Set $R_{K'} = \{a \in A^\star(K')\ ,$
$$u_{\tau_{K'}}(a) = \max_{b \in A^\star(K')} u_{\tau_{K'}}(b) \Big\}$$
- For k iterating from $K' - 1$ downwards to 1,
  - For any $a \in A$ and $a^\star \in R_{k+1}$, define
  $$F_k(a, a^\star) = \begin{cases} a & \text{if } \forall i \in A \setminus \{\tau_k\} \text{ either} \\ & u_i(a) \geq \rho(k) \text{ or } u_i(a) \geq u_i(a^\star) \\ a^\star & \text{otherwise} \end{cases}$$
  - Set
  $$R_k = \bigcup_{a^\star \in R_{k+1}} \{F_k(a, a^\star)\ ,\ a \in A \text{ and}$$
  $$u_{\tau_k}(F_k(a, a^\star)) = \max_{b \in A} u_{\tau_k}(F_k(b, a^\star)) \Big\}$$

The set of options resulting from that algorithm is R_K.

Algorithm ***Algo4*** has a polynomial complexity. More precisely, the maximal number of operations that Algorithm ***Algo4*** can perform is of order $K'\, p^2\, (2\, n + p)$.

Proposition 4 *Under the assumptions of Lemma 2, R_K resulting from Algorithm **Algo4** is the set of possible outcomes of the SPEs of protocol **P1**.*

The following axiom assumes that any player is periodically a proposer.

Axiom **Periodical Proposers (PP)**. There exists $T \in \mathbb{N}$ such that for any $k \in \mathbb{N}_*$ and any $i \in N$, there exists $j \in \{k, k+1, \dots, k+T\}$ such that $\tau_j = i$.

Let $A^\star = \left\{ a \in A\ ,\ \min_{i \in N} u_i(a) = \max_{b \in A} \min_{i \in N} u_i(b) \right\}$ be the maximin solution.

We consider now families $\{\rho_i^\delta\}_\delta$ of functions ρ_i parametrized by a value $\delta \in (0, 1)$ such that

$$\forall k \in \mathbb{N}\ ,\ \left|\rho_i^\delta(k) - \rho_i^\delta(k+1)\right| \leq c\, \delta \qquad (1)$$

For these functions, one can attain any arbitrarily small differences between two successive thresholds. Note that the upper bound for the maximal time of an end to the protocol depends on δ.

We show that under previous condition, equilibria correspond to the maximin solutions.

Proposition 5 *Let $\{\rho_i^\delta\}_\delta$ be families of functions satisfying (1), **(PP)** and **(T)** such that $\rho_1^\delta = \cdots = \rho_m^\delta$ and $\rho_1^\delta(0) = \cdots = \rho_m^\delta(0) = 1$. Then there exists $\Lambda \in (0, 1)$ such that for any $\delta \in (0, \Lambda)$, the SPEs of protocol **P1** are elements of $A^\star$.*

The idea of the proof is that if δ is small enough, then all players will be proposers at least once while ρ^δ lies in-between $\underline{u}(A \setminus A^\star)$ and $\underline{u}(A^\star)$. Before those iterations, there is always at least one player that is not forced to accept a proposal. For those iterations, the only proposal for which all players are forced to accept are $A^\star$. After those iterations, more options can be necessarily accepted but such options are disadvantageous for at least one player. This player could have avoided getting this bad option by just proposing any element of $A^\star$ during his turn in previous iterations.

4 Strategy proofness of Protocol *P1*

The Gibbard-Satterthwaite theorem [7, 15] shows that there is no democratic voting rule that is not manipulable. A voting rule is manipulable when the outcome of the vote is better to a DM if he announces preferences that are not his true ones, compared to when he reveals his true preferences. This definition of manipulation can be extended to consensus mechanisms.

Definition 6 *A consensus mechanism M is* subject to manipulation *if*

$$\exists i \in N\ ,\ \exists u_i' \in \mathcal{U}(A) \quad \textit{such that} \quad u_i\left(M(u_i', u_{-i}\right) > u_i\left(M(u_1, \dots, u_n)\right)$$

In this definition u_i represents the true utility of i, and u_i' is what DM i announces.

If $a^\star$ is the best option of player i (according to his true utility u_i), and if player i announces the following utility

$$\tilde{u}_i(a) = \begin{cases} \varepsilon & \text{if } a \neq a^\star \\ 1 & \text{if } a = a^\star \end{cases} \qquad (2)$$

then for ε small enough, $a^\star$ is the outcomes of all SPE of protocol ***P2***. As a result, protocol ***P2*** is subject very much to manipulation. An easy manipulation for a DM consists thus to over-estimate his best options and under-estimate the other ones. This strategy does not require to know the private information of the other DMs.

This drawback may be see as acceptable in the context of cooperation. However, even if the DMs accept to cooperate, they may wish to adopt a strategy that will favor their personal interest. So they may exaggerate their preferences or change them a little bit.

4.1 Detecting exaggerations

Protocol ***P1*** assumes that utility functions u_i represent the true preferences of the DMs and that these utility functions are *commensurate* among the DMs. This implies that a score say 0.6 given by one DM has the same meaning that the same score 0.6 for another DM. This assumption is very hard to obtain. Multi-Criteria Decision Analysis and Measurement Theory provide a solution to this problem [2, 8]. The idea is to construct utility function u_i as an interval scale. This is a numerical representation of the preferences of the DM in which the notion of difference makes sense. This means that the difference of satisfaction between options a and b is $\frac{u_i(b)-u_i(a)}{u_i(d)-u_i(c)}$ times more important than the difference of satisfaction between options c and d. An interval scale is given up to an affine transformation. The idea of the Macbeth approach is then to identify in A two references elements and to fix entirely the interval scale by fixing the utility for these two particular elements. These two elements are chosen so that they have the same meaning throughout the DM. The first element corresponds to an option $\mathbf{0}_i \in A$ that is considered as completely unacceptable to DM i. The second element is an option $\mathbf{1}_i \in A$ that is considered as completely satisfactory to DM i. Elements $\mathbf{0}_1, \ldots, \mathbf{0}_n$ have the same meaning, and so do elements $\mathbf{1}_1, \ldots, \mathbf{1}_n$. We furthermore set $u_i(\mathbf{0}_i) = 0$ and $u_i(\mathbf{1}_i) = 1$ for all $i \in N$.

Previous methodology is designed to construct utility functions that really represent the preferences of the DM and to ensure commensurateness between the DMs. Note that it is possible that each utility function u_i is constructed from multiple criteria [2, 8].

It is always possible that a DM has sheeted during the elicitation of his utility function. This means that the utility function he obtains is somehow in-between his true one and the one described in (2). We want to stress the fact that it is possible to some extend to guess whether a DM has exaggerated his preferences.

An easy way to exaggerate his preferences is to reduce the support of the utility function, making it sharper. Let us show this in a simple case. Assume that the options are described by one attribute and are ranked according to this attribute in the following order $a_1, a_2, \ldots, a_p$. We assume furthermore that the options are uniformly spread over the values of the attribute. Then $\Delta_l u_i := u_i(a_{l+1}) - u_i(a_l)$ is proportional to the derivative of u_i at the value of the attribute corresponding to a_l. According to previous remark, the more DM i exaggerates his preferences, the sharper u_i in some parts, and thus the less uniform $\Delta_l u_i$ over l. It appears thus that a measure of uniformity of $\Delta_l u_i$ can assess the degree of exaggeration of DM i. Since the Shanon entropy is classically used to measure such uniformity, we make the following definition

$$\mathcal{W}_i := \sum_{l \in \{1,\ldots,p-1\}} -\Delta_l u_i \, \log(\Delta_l u_i) \ .$$

Comparing the value of $\mathcal{W}_i$ for all DMs, one can detect the more exaggerating ones, i.e. the ones with the smallest values of $\mathcal{W}_i$. They can be asked to revise their utility function, in the spirit of [18, 4].

4.2 On the manipulation of protocol *P1*

We have seen that when commensurateness among DMs is assumed, the protocol is very sensitive to manipulation. It is not so easy to renormalize cardinal utility functions in order to obtain this property. In (2), a DM can exaggerate the difference of preferences and thus of utility between two options. One way to get rid of this problem is to construct a utility function that is not based on difference of intensities in the preferences but just on an ordinal information which is the ranking of the options according to a DM.

More precisely, we consider an isomorphism τ_i from A onto the integer set $\{1, \ldots, p\}$. The set of all isomorphisms from A onto the integer set $\{1, \ldots, p\}$ is denoted by $\mathcal{I}(A)$. τ_i is assumed to represent the ordinal preferences of DM i over A. This representation is valid iff the preference relation of DM i is a total order (i.e. having no ex equo). Then $a \in A$ is preferred to $b \in A$ according to DM $i \in N$ if $\tau_i(a) > \tau_i(b)$. Then we define $u_i(a) := \frac{1}{p}\, \tau_i(a)$. Set $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{I}(A)^n$.

Remark that τ cannot easily be determined when A contains many elements, for instance when A is a combinatorial space.

Each DM has thus to define the ordering τ_i he wants to be applied in the game. The overall problem can thus be seen as a two-stage game. Firstly each DM has to provide his ranking on the set A of options. This corresponds to a distortion game [12]. Secondly, the DMs enters in the protocol with fixed utility functions defined from τ. One can assume that the solutions to the second game are the Maximin solutions $A^\star(\tau)$.

A vector $\tau^\star \in \mathcal{I}(A)^n$ is a Nash equilibrium of the first game iff for all $i \in N$ and all $\tau_i \in \mathcal{I}(A)$,

$$u_i(A^\star(\tau^\star)) \geq u_i(A^\star(\tau_i, \tau^\star_{-i})) \ .$$

Set for $S \subseteq N$, $\tau^\wedge_S(a) := \min_{k\in S} \tau_k(a)$, and, for $i \in N$, $\tau^\wedge_{-i}(a) := \tau^\wedge_{N\setminus\{i\}}(a)$. Let

$$A^\star(\tau) := \left\{ a \in A \, , \tau^\wedge_N(a) = \max_{b\in A} \tau^\wedge_N(b) \right\}$$

Lemma 7 *Let $\tau_{N\setminus\{i\}} \in \mathcal{I}(A)^{n-1}$ and $a \in A$. Then there exists $\tau_i \in \mathcal{I}(A)$ such that $A^\star(\tau) = \{a\}$ iff*

$$\left|\left\{ b \in A \setminus \{a\} \, , \ \tau^\wedge_{-i}(b) \geq \tau^\wedge_{-i}(a) \right\}\right| < \tau^\wedge_{-i}(a) \quad \text{and } \tau^\wedge_{-i}(a) > 1 \tag{3}$$

Let

$$A_i(\tau_{-i}) := \{ a \in A \, , \ \ (3) \text{ is satisfied} \} \ .$$

Corollary 8 *Let $\tau_{N\setminus\{i\}} \in \mathcal{I}(A)^{n-1}$. Set $\tilde{D}_k = \{a \in A \, , \ \tau^\wedge_{-i}(a) \geq k\}$.*

There exists $\tau_i \in \mathcal{I}(A)$ such that $A^\star(\tau)$ is reduced to a singleton iff there exists $k \in \{2, \ldots, p\}$ such that

$$|\tilde{D}_k| \leq k \ . \tag{4}$$

Let K be the smallest integer k satisfying (4). Then $A_i(\tau_{-i}) = \tilde{D}_K$. Moreover, $|A_i(\tau_{-i})| \leq \lceil \frac{p}{2} \rceil$, where $\lceil r \rceil$ denotes the smallest integer larger or equal to r.

Lemma 9 *$A^\star(\tau)$ is reduced to a singleton iff there exists $a \in \tilde{D}_K$ such that*

$$\begin{aligned} &\forall b \in D_a \, , \ \tau_i(b) < \min(\tau_i(a), \tau^\wedge_{-i}(a)) \\ &\text{and either } \tau_i(a) \geq \tau^\wedge_{-i}(a) \text{ or } \forall b \in A \setminus (D_a \cup \{a\}) \, , \\ &\qquad \min(\tau_i(b), \tau^\wedge_{-i}(b)) < \tau_i(a) \end{aligned} \tag{5}$$

Assume that $A_i(\tau_{-i}) \neq \emptyset$. Then player i cannot act in such a way to obtain an option not in $A_i(\tau_{-i})$. Consider the option $a^\star(i)$ of $A_i(\tau_{-i})$ that has the largest value of τ_i. This is the best option that DM i can get. If the conditions of Lemma 7 are not satisfied for option $a^\star_i$, then DM i can announce another $\tilde{\tau}_i$. He can set $\tilde{\tau}_i(a^\star(i)) = \max(\tau_i(a^\star(i)), \tau^\wedge_{-i}(a^\star(i)))$. Then (5) becomes

$$\forall b \in D_{a^\star(i)} \, , \ u_i(b) < u_i(a^\star(i)) \ .$$

Since $|D_{a^\star(i)}| < \tau_i(a^\star(i))$, it is always possible to define $\tilde{\tau}_i(b)$ for $b \in D_{a^\star(i)}$ such that previous relation holds.

We study here in the situation of all DMs except one are giving their true preferences, and we investigate the interest for the last player to lie on his true preferences. The major difference between the cardinal case and the ordinal case is that a DM cannot obtain his preferred option by choosing strategy (2). In the ordinal case, he can only force to obtain one of the most preferred options for the remaining DMs. This set is some kind of consensus of the remaining DMs.

5 A protocol based on coalitions of players

5.1 Protocol

In Protocol ***P1***, each individual DM is actor of the protocol. It is well-known that individual DMs become stronger when they group themselves. This lead to the case where the actors correspond to a coalition structure, that is a partition of N. One can also think of a generalization of that, considering all coalitions that can form. Let us denote by $\mathcal{W} \subseteq 2^N$ the set of admissible coalitions. One can think of the set of *winning coalitions* in the sense of a voting rule. We are thus interested in a protocol where the actors that give their opinion are in fact all coalitions in $\mathcal{W}$.

The second protocol is similar to ***P1***, except that the actors are coalitions of $\mathcal{W}$ rather than individual DMs. The proposers and responders are elements of $\mathcal{W}$. The sequence of proposers is supposed to be known in advance and is denoted by $\Pi_1, \Pi_2, \ldots$

Protocol ***P2***. At iteration $k \in \mathbb{N}_*$:

- Coalition Π_k makes an offer $a \in A$ (a can already be proposed earlier).
- All other coalitions in $\mathcal{W} \setminus \{\Pi_k\}$ say *Yes* or *No*. However, a coalition S must accept a if $e(S, a) \geq \rho_S(k)$.
- If at least one coalition rejects the offer, we go to the next iteration. Otherwise, the option is chosen and the protocol ends.

Lemma 10 *Under* **(T)***, protocol* ***P2*** *has at least one Subgame Perfect Nash Equilibrium (SPE).*

5.2 A generalization of Minimum Regret Solution

The idea of the maximin solution is to select the option for minimize the dissatisfaction of the player that enjoys the less the option. The maximin solution is the option (or the set of options) a such that

$$\min_{i\in N} \ u_i(a) = \max_{a'\in A} \ \min_{i\in N} \ u_i(a') \ .$$

We will see in a second that the Nucleolus proposes another definition of dissatisfaction. Instead of looking at the worse score of an option over the players, the idea of the Nucleolus is to look at the difference between what a set of players get with an option and what they could get at best.

The bargaining sets are based on the notion of *justified objection*. An objection is justified whenever there is no counterobjection. One sees that this definition is purely qualitative. There is indeed no notion of intensity of an objection or a counterobjection. This may explain why one cannot obtain nonemptyness in the general case [17]. In order to avoid the consequences of Arrow's theorem, one may think of quantitative concepts, such as the *Nucleolus*. This notion is classically based on the concept of *excess*. For a classical TU game, the excess for a given coalition and an imputation is the difference between what the game will provide for this coalition and what the imputation promises to give. In our case, the excess is defined by

$$e(S,a) = \max_{b \in A} \sum_{i \in S} u_i(b) - \sum_{i \in S} u_i(a) \ .$$

$e(S,a)$ is the difference between the maximal possible total satisfaction degree to players S and what they get with a. $e(S,a)$ cannot be negative. This notion of excess generalizes the notion of *regret* known in Social Choice [1]. In Social Choice, N represents the society, and $\sum_{i \in N} u_i(a)$ represents the total worth represented by a on the whole society. The option that is *best for society* is thus the option a that maximizes $\sum_{i \in N} u_i(a)$. The regret associated to an option a is then defined as $e(N,a)$. In Social Choice, the *minimum regret* option procedure corresponds to selecting the option that minimizes regret $e(N,a)$.

A strictly positive excess $e(S,a)$ may be interpreted as a dissatisfaction of the coalition S when faced with the proposal a. Hence, vector $\{e(S,a)\}_{S \subseteq N}$ measures the dissatisfaction of the subcoalitions of N about a. Instead of expressing dissatisfaction as an objection that one player can make to another one, as in the bargaining set, another option is to consider the alternatives a whose highest complaint $\max_{S \subseteq N} e(S,a)$ is the smallest one. Define thus $\succeq_{\max}$ by $(e(S,b))_{S \subseteq N} \succeq_{\max} (e(S,a))_{S \subseteq N}$ iff $\max_{S \subseteq N} e(S,b) \geq \max_{S \subseteq N} e(S,a)$.

A first idea is to select the option that minimizes the The leads to the maximin solution on the excess :

$$\mathcal{K} = \left\{ a \in A \ , \ \max_{S \subseteq N} e(S,a) = \min_{b \in A} \max_{S \subseteq N} e(S,b) \right\} \ ,$$

and

$$\mathcal{K}_{\mathcal{W}} = \left\{ a \in A \ , \ \max_{S \in \mathcal{W}} e(S,a) = \min_{b \in A} \max_{S \in \mathcal{W}} e(S,b) \right\} \ ,$$

where $\mathcal{W}$ is a subset of 2^N.

We will then show that under some assumptions, all equilibria of this protocol are exactly the elements of the maximin set $\mathcal{K}_{\mathcal{W}}$ on the excess. This gives a theoretical characterization of $\mathcal{K}_{\mathcal{W}}$. Since the Nucleolus $\mathcal{N}_{\mathcal{W}}$ is a refinement of $\mathcal{K}_{\mathcal{W}}$, it also justifies this result. We want to stress that Protocol ***P2*** does not pretend to be realistic in practice. It serves as a bargaining characterization of $\mathcal{K}_{\mathcal{W}}$.

Lemma 11 *Let $\{\rho_S^\delta\}_\delta$ be families of functions satisfying (1),* **(PP)** *and* **(T)** *such that ρ_S^δ does not depend on coalition S, and $\rho_S^\delta(0) = 1$ for all coalitions S. Then there exists $\Delta \in (0,1)$ such that for any $\delta \in (0, \Delta)$, the SPEs of protocol **P2** are the elements of $\mathcal{K}_{\mathcal{W}}$.*

5.3 Nucleolus

It is well-known that the max ordering is not quite discriminative since the two payoff vector are judged similar as soon as their maximal value is the same. It is possible to refine this $\succeq_{\max}$ ordering, in such a way to be more discriminative. We say that an ordering $\succeq'$ is a refinement of $\succeq_{\max}$ if $x \succeq' y$ whenever $x \succeq_{\max} y$, $x \succ' y$ whenever $x \succ_{\max} y$, and there exists z, t such that $z \succ' t$ and $z \sim_{\max} t$. The *leximax* ordering defined below is a refinement of the max ordering.

Define the *lexicographic* ordering $\succeq_{lex}$ on vectors in $\mathbb{R}^d$ (for some $m \in \mathbb{N}$) by : $x \succeq_{lex} y$ iff

$$\exists p \in \{1, \ldots, m\} \ , \ x_j = y_j \text{ for all } j \in \{1, \ldots, p\} \text{ and } x_{p+1} > y_{p+1} \ .$$

Let $x, y \in \mathbb{R}^d$ with τ, π permutations on $\{1, \ldots, d\}$ such that $x_{\tau(1)} \geq \cdots \geq x_{\tau(d)}$ and $y_{\pi(1)} \geq \cdots \geq y_{\pi(d)}$. The *leximax* ordering is defined by

$$x \succeq_{leximax} y \ \text{ iff } \ (x_{\tau(1)}, \ldots, x_{\tau(d)}) \succeq_{lex} (y_{\pi(1)}, \ldots, y_{\pi(d)}) \ .$$

Define $\succ_{\mathcal{N}}$ by

$$a \succeq_{\mathcal{N}} b \quad \Longleftrightarrow \quad (e(S,b))_{S \subseteq N} \succeq_{leximax} (e(S,a))_{S \subseteq N}$$

The *prenucleolus* set is the option of A such that the associated vector $(e(S,a))_{S \subseteq N}$ in $\mathbb{R}^{2^N}$ is the smallest element in the leximax sense [13] :

$$\mathcal{N} = \{ a \in A \ , \ \forall b \in A \ \ a \succeq_{\mathcal{N}} b \}$$

Lemma 12 $\mathcal{N} \neq \emptyset$.

Example 13 *Consider 4 options and 3 players.*

	Option a	*Option b*	*Option c*
Player 1	1	0.4	0.7
Player 2	0.5	0.3	0.2
Player 3	0.2	0.3	0.7

The excess values are given in the following array.

$e(\cdot,\cdot)$	*Option a*	*Option b*	*Option c*
$\{1\}$	0	0.6	0.3
$\{2\}$	0	0.2	0.3
$\{3\}$	0.5	0.4	0
$\{1,2\}$	0	0.8	0.6
$\{1,3\}$	0.3	0.8	0
$\{2,3\}$	0.2	0.3	0
$\{1,2,3\}$	0	0.7	0.1

One has $\mathcal{N} = \{a\}$ *whereas* b *is the maximin solution. The choice of option* a *has much less complaint from players and groups of players than the selection of* b.

Lemma 14 *If* $a, b \in \mathcal{N}$ *then*

$$\sum_{i\in N} u_i(a) = \sum_{i\in N} u_i(b) \ .$$

In the general case, the Nucleolus is not necessarily reduced to a singleton, as shown in the following result.

Lemma 15 *Let* $a, b \in A$ *such that* $\sum_{i\in N} u_i(a) = \sum_{i\in N} u_i(b)$, *and for all* $c \in A \setminus \{a, b\}$ *and all* $i \in N$,

$$u_i(c) < u_i(a) \quad \text{and} \quad u_i(c) < u_i(b) \ .$$

Then

$$\mathcal{N} = \{a, b\} \ .$$

However, as the following example shows, the elements of the Nucleolus do not necessarily have the largest value of $\sum_{i\in N} u_i(a)$, or equivalently the smallest value of the overall excess $\sum_{S\subseteq N} e(S, a)$. This comes from the fact that the Nucleolus is derived from the max ordering.

Example 16 *Consider* 4 *options and* 3 *players.*

	a	b	c	d
Player 1	1	0.5	0.5	0.6
Player 2	0	0.5	0	0.1
Player 3	0.2	0.2	0.7	0.3

The excess values are given in the following array.

$e(\cdot,\cdot)$	a	b	c	d
$\{1\}$	0	0.5	0.5	0.4
$\{2\}$	0.5	0	0.5	0.4
$\{3\}$	0.5	0.5	0	0.4
$\{1,2\}$	0	0	0.5	0.3
$\{1,3\}$	0	0.5	0	0.3
$\{2,3\}$	0.5	0	0	0.3
$\{1,2,3\}$	0	0	0	0.2

One has $\mathcal{N} = \{d\}$. *However,*

$$\sum_{S\subseteq N} e(S, d) = 2.3 > \sum_{S\subseteq N} e(S, a) = \sum_{S\subseteq N} e(S, b) = \sum_{S\subseteq N} e(S, c) = 1.5$$

So, one sees that the overall excess on all possible coalitions is much worse for d *than for* a, b *and* c.

It is possible to restrict the definition of the Nucleolus to winning coalitions. This means that one does not care about the excess for losing coalitions.

$$\mathcal{N}_{\mathcal{W}} = \{a \in A\ ,\ \ \forall b \in A \quad (e(S, b))_{S\in\mathcal{W}} \succeq_{leximax} (e(S, a))_{S\in\mathcal{W}}\} \ .$$

Following Example 16, the overall excess $\sum_{S\in\mathcal{W}} e(S, a)$ is an interesting measure together with the leximax ordering. To this end, the leximax ordering has to be transformed into a utility model. It is well-known that the *leximax* ordering cannot in the general case be described by a utility model, i.e.

$$x \succeq_{leximax} y \quad \Longleftrightarrow \quad F_{leximax}(x) \geq F_{leximax}(y) \ .$$

This is possible when the scale in discrete, which is not our case. However, we replace the leximax ordering by that induced by such function $F_{leximax}$. This gives a coarsening of the leximax ordering but it is still a refinement of the max ordering.

Define thus for $\tau \in [0, 1]$ for $x : \mathcal{W} \to \mathbb{R}$

$$G_\tau(x) = \tau \sum_{S\in\mathcal{W}} x(S) + (1 - \tau)\, F_{leximax}(x) \ .$$

Function G_τ describes a tradeoff between the largest excess and the mean excess. Then define

$$\mathcal{N}^\star_{\mathcal{W}} = \{a \in A\ ,\ \ \forall b \in A \quad G_\tau\left(\{e(S, b)\}_{S\in\mathcal{W}}\right) \geq G_\tau\left(\{e(S, a)\}_{S\in\mathcal{W}}\right)\} \ .$$

Example 17 *Consider* 3 *options and* 3 *players.*

	Option a	*Option b*	*Option c*
Player 1	0.5	0	0
Player 2	0.1	0.5	0.7
Player 3	0.5	0.5	0

It is easy to seen that $a \succ_{\mathcal{N}} b$. *Assume now that option* c *is withdrawn. Removing* c *from* A *leads to the reverse relation* $b \succ_{\mathcal{N}} a$. *We conclude that the presence of* c *influences the preference between* a *and* b. *Hence the Nucleolus does not satisfy the* Independence to Irrelevant Alternatives *axiom [1].*

6 Conclusion

We have introduced in Section 3 a negotiation protocol that is mainly characterized by an acceptance threshold that obliges the DMs to accept an offer having a better utility than that threshold. In order to predict the outcome of this protocol, we have studied its SPEs. In the case where the thresholds decreases slowly enough, we show that the SPEs correspond to the maximin solution. In the general case, we have designed an algorithm that compute the outcomes of all possible SPEs.

We notice in Section 4 that the protocol is subject to manipulation by the DMs. Two possible proposals to reduce this sensitivity are made. The first one consists in the detection of a possible distortion of the initial preferences. The second one proposes an ordinal version of the protocol, which reduces the influence of manipulation.

Previous protocol has been generalized in Section 5 allowing coalitions to interact. This leads to a solution concept that generalizes the notion of minimum regret and that seems to be new.

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