Lattice-Valued Possibilistic Entropy Functions

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Abstract

Lattice-valued entropy functions defined by a lattice-valued possibilistic distribution π on a space Ω are defined as the expected value (in the sense of Sugeno integral) of the complement of the value $\pi(\omega)$ with ω ranging over Ω. The analysis is done in parallel for two alternative interpretations of the notion of complement in the complete lattice in question. Supposing that this complete lattice is completely distributive in the defined sense, the entropy values defined by independent products of finite sequences of lattice-valued possibilistic distributions are proved to be defined by the supremum value of the entropies defined by particular distributions.

Keywords: lattice-valued possibilistic distribution, possibilistically independent product of distributions, possibilistic entropy functions.

1 Introduction and Motivation

Since the earliest investigations in the field of possibilistic (possibility) theory and measures as conceived by L. A. Zadeh in 1978 [5], possibilistic measures have been considered as a qualitatively different alternative tool for uncertainty quantification and processing, if compared with those offered by additive measures and probability theory, but the notions introduced and results achieved by the classical probabilistic approach have been continually used as an inspiration for a further development of possibilistic theory and measures (for our purposes, first of all [1] is worth being explicitly mentioned).

Within the framework of standard (Shannon) information theory, based on Kolmogorov axiomatic probability theory, the notion of *entropy* plays the fundamental role, as it enables to quantify the amount of uncertainty (in the sense of randomness) contained or hidden in a probability distribution over a nonempty space of random events, in other sense, the (expected) amount of information obtained when realizing a random sample from this distribution. A great number of important and interesting results dealing with information encoding and transmission and with statistical decision making under uncertainty have been achieved when applying the notion of information. So, it seems quite reasonable to seek for an alternative modification of the notion of entropy applicable to possibilistic measures. For real-valued possibilistic measures some rather elementary ideas and results can be found in [4], the case of non-numerical and, in particular, lattice-valued possibilistic measures will be, very briefly, discussed below. Let us sketch, very shortly, the probabilistic version of the notion of entropy function (cf. [3] or any elementary textbook or monograph on information theory).

Restricting ourselves to the most simple case, let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be a finite space and let $P : \Omega \rightarrow [0,1]$ be a probability distribution on Ω, hence, $\sum_{i=1}^{n} P(\omega_i) = 1$. *(Shannon) entropy* $I(P)$ of P is defined by

$$I(P) = \sum_{i=1}^{n} (lg(1/P(\omega_i))P(\omega_i) = -\sum_{i=1}^{n} (lg\, P(\omega_i))P(\omega_i) \tag{1.1}$$

setting $0\, lg\, 0 = 0$ and applying, as a rule, the logarithm to the base 2, so that $I(P)$ is the expected value of the random variable $1/P(\cdot)$. Within the framework of possibilistic measures the normalized Sugeno integral will play the role of expected value, so that an alternative entropy function $I^*(P)$, defined by

$$I^*(P) = \sum_{i=1}^{n} (1 - P(\omega_i))P(\omega_i) = 1 - \sum_{i=1}^{n} (P(\omega_i))^2, \tag{1.2}$$

will serve as an inspiration. Both $I(P)$ and $I^*(P)$ take their maximum values ($lg\, n$ for $I(P)$ and $1-(1/n)$ for $I^*(P)$) iff $P(\omega_i) = 1/n$ for each $i \leq n$ (the uniform probability distribution on Ω), and take their minimum value 0 for both $I(P)$ and $I^*(P)$ iff $P(\omega_{i_0}) = 1$ for one $i_0 \leq n$ (hence, $P(\omega_i) = 0$ for each $i \leq n, i \neq i_0$).

2 Lattice-Valued Possiblistic Distributions

Partially ordered set (poset) is a pair $\mathcal{T} = \langle T, \leq \rangle$, where T is a non-empty set and $\leq$ is a partial ordering on T, i.e., $\leq$ is a reflexive, antisymmetric and transitive binary relation on T (subset of $T \times T$, in set-theoretic terms). Let $\bigvee, \vee (\bigwedge, \wedge$, resp.) denote the supremum (infimum, resp.) operation defined by $\leq$ in $\mathcal{T}$. Poset $\mathcal{T}$ is called *complete lattice*, if for each nonempty subset $A \subset T$ its supremum $\bigvee A (= \bigvee_{t \in A} t)$ and infimum $\bigwedge A (= \bigwedge_{t \in A} t)$ are defined. The *zero (unit,* resp.) element of the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is denoted by $\oslash_{\mathcal{T}}$ ($\mathbf{1}_{\mathcal{T}}$, resp.) and defined by $\oslash_{\mathcal{T}} = \bigwedge T$ ($\mathbf{1}_{\mathcal{T}} = \bigvee T$, resp.). By convention we set $\bigvee \emptyset = \oslash_{\mathcal{T}}$ and $\bigwedge \emptyset = \mathbf{1}_{\mathcal{T}}$ for the empty subset of T.

A complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is called *distributive,* if the relation $s \wedge (t_1 \vee t_2) = (s \wedge t_1) \vee (s \wedge t_2)$ holds for each $s, t_1, t_2 \in T$ and $\mathcal{T}$ is called *completely distributive,* if the relation $s \wedge (\bigvee A) = \bigvee_{t \in A} (s \wedge t)$ holds for each $s \in T$ and $\emptyset \neq A \subset T$. The inequality $s \wedge (\bigvee A) \geq \bigvee_{t \in A} (s \wedge t)$ obviously holds for each poset $\mathcal{T}$, but the inverse inequality need not hold in general. E.g., the complete lattice $\mathcal{T}_0 = \langle \{t_0, t_1, t_2, t_3, t_4\}, \leq \rangle$ such that $t_0 < t_i < t_4$ holds for each $i = 1, 2, 3$, but no $t_i, t_j, i, j = 1, 2, 3, i \neq j$, are comparable by $\leq$, is not distributive. Both the most often used structures for uncertainty degrees, i.e., the unit interval of real numbers equipped by their standard ordering, as well as the space of all subsets of a nonempty set X partially ordered by set inclusion, can be easily seen to define completely distributive complete lattices.

Contrary to the operation $1 - x$ for $x \in [0, 1]$ or the set complement $X - A$ for $A \subset X$ there is no general primary operation of complement in complete lattices. Let us define, for our purposes, the $\mathcal{T}$-complement t^C for each $t \in T$, setting

$$t^C = \bigvee \{s \in T : s \wedge t = \oslash_{\mathcal{T}}\}. \tag{2.1}$$

In particular, $\oslash_{\mathcal{T}}^C = \mathbf{1}_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}}^C = \oslash_{\mathcal{T}}$, for $\mathcal{T} = \langle [0, 1], \leq \rangle$ we obtain $0^C = 1$ and $x^C = 0$ for each $0 < x \leq 1$, and for $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$ $\mathcal{T}$-complement agrees with the standard set-complement operation.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty space. A mapping π which takes Ω into T is called $\mathcal{T}$*-(valued) possibilistic distribution* on Ω, if $\bigvee_{\omega \in \Omega} \pi(\omega) = \mathbf{1}_{\mathcal{T}}$ holds, hence, if π is a $\mathcal{T}$-valued normalized fuzzy subset of Ω as introduced by J. A. Goguen in [2]. The $\mathcal{T}$*-possibilistic measure* Π induced by π on the power-set $\mathcal{P}(\Omega)$ is defined by $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$, hence, $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ and $\Pi(\emptyset) = \oslash_{\mathcal{T}}$ by convention for the empty subset of Ω.

3 Lattice-Valued Possibilistic Entropy Functions

In what follows, we will need a mathematical tool for integration of lattice-valued mappings with respect to lattice-valued possibilistic measures.

As the first attempt, for these sakes the notion of Sugeno integral as analyzed and applied in [1] with the minimum operation on T taken as the particular case of t-norm seems to be more or less adequate and the limited scope of this contribution does not offer a space for a more detailed discussion on this matter. Nevertheless, a modification of notions and results to be introduced below to the case of a general t-norm processed by a tool of integration more powerful than the minimum-based Sugeno integral seems to be, beyond any doubts, worth being pursued in more detail when going on with the research dealing with lattice-valued possibilistic entropy functions.

Definition 3.1 Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a non-empty space, let π be a $\mathcal{T}$-possibilistic distribution on Ω with Π denoting the $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$ induced by π, let f be a mapping which takes Ω into T. The value

$$\oint f d\Pi = \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] \tag{3.1}$$

is called the *(Sugeno) integral of the mapping f* over the space Ω, with respect to the possibilistic measure Π on $\mathcal{P}(\Omega)$ and with infimum operation $\wedge$ taken as the t-norm on T.

As a matter of fact, as analyzed and proved in [1], under our setting the value $\oint f d\Pi$ is always defined and the relation

$$\oint f d\Pi = \bigvee_{\omega \in \Omega} (f(\omega) \wedge \pi(\omega)) \tag{3.2}$$

holds. In order to recall the role of integrals as expected values of variables charged by uncertainty we will often write $Ef d\Pi$ or $Ef(\cdot)d\Pi$ instead of $\oint f d\Pi$.

When aiming to modify the definition of entropy function $I^*(P)$, given by (1.2), to the case of lattice-valued possibilistic distributions, the problem arises how to replace the substraction operation $1-x$. In the case of probability distribution P over finite or countable space Ω the identity $1-P(\omega) = P_*(\Omega - \{\omega\}) = \sum_{\omega_1\in\Omega,\omega_1\neq\omega} P(\omega_1)$ is obvious, but in the case of possibilistic distributions (real-valued as well as lattice-valued ones in general) the relation $1-\pi(\omega) = \Pi(\Omega - \{\omega\})$ or $(\pi(\omega))^C = \Pi(\Omega - \{\omega\})$ need not hold in general. Aiming to take in consideration both the cases in parallel, we arrive at the following definition.

Definition 3.2 Let $\mathcal{T} = \langle T, \leq\rangle$ be a complete lattice, let π be a $\mathcal{T}$-possibilistic distribution on a nonempty space Ω. (1)-*entropy* of π is denoted as $I_1(\pi)$ and defined by

$$I_1(\pi) = E(\pi(\omega))^C d\Pi = \bigvee_{\omega\in\Omega} (((\pi(\omega))^C \wedge \pi(\omega)), \quad (3.3)$$

(2)-*entropy* of π is denoted as $I_2(\pi)$ and defined by

$$I_2(\pi) = E((\Pi(\Omega - \{\omega\}))d\Pi = \bigvee_{\omega\in\Omega} ((\Pi(\Omega - \{\omega\}))\wedge\pi(\omega)). \quad (3.4)$$

Both the values $I_1(\pi)$ and $I_2(\pi)$ are evidently defined for each $\mathcal{T}$-possibilistic distribution π on Ω.

Theorem 3.1 Let $\mathcal{T} = \langle T, \leq\rangle$ be a distributive complete lattice, let π be a $\mathcal{T}$-possibilistic distribution on a nonempty space Ω. Then the inequality $I_1(\pi) \leq I_2(\pi)$ holds. In particular, if $\mathcal{T}$ is completely distributive, then $I_1(\pi) = \emptyset_{\mathcal{T}}$, so that the inequality $I_1(\pi) \leq I_2(\pi)$ follows trivially.

Proof. Let $s \in T$ and $\Omega \in \Omega$ be such that $s \wedge \pi(\omega) = \emptyset_{\mathcal{T}}$ holds. Due to the assumed distributivity of $\mathcal{T}$ we obtain that

$$\begin{aligned} s &= s \wedge \mathbf{1}_{\mathcal{T}} = s \wedge \left(\bigvee_{\omega_1\in\Omega} \pi(\omega_1)\right) = \\ &= s \wedge \left(\left(\bigvee_{\omega_1\in\Omega,\omega_1\neq\omega} \pi(\omega_1)\right) \vee \pi(\omega)\right) = \\ &= \left(s \wedge \bigvee_{\omega_1\in\Omega,\omega_1\neq\omega} \pi(\omega_1)\right) \vee (s \wedge \pi(\omega)) = \\ &= s \wedge \left(\bigvee_{\omega_1\in\Omega,\omega_1\neq\omega} \pi(\omega_1)\right), \end{aligned} \quad (3.5)$$

so that the inequalities $s \leq \bigvee_{\omega_1\in\Omega,\omega_1\neq\omega} \pi(\omega_1)$ and

$$\begin{aligned} (\pi(\omega))^C &= \bigvee\{s \in T : s \wedge \pi(\omega) = \emptyset_{\mathcal{T}}\} \leq \\ &\leq \bigvee_{\omega_1\in\Omega,\omega_1\neq\omega} \pi(\omega_1) = \Pi(\Omega - \{\omega\}) \end{aligned} \quad (3.6)$$

are valid. Hence, the inequality

$$\begin{aligned} I_1(\pi) &= \bigvee_{\omega\in\Omega} ((\pi(\omega))^C \wedge \pi(\omega)) \leq \\ &\leq \bigvee_{\omega\in\Omega} ((\Pi(\Omega - \{\omega\}) \wedge \pi(\omega)) = I_2(\pi) \end{aligned} \quad (3.7)$$

results. If $\mathcal{T}$ is completely distributive, then for each $t \in T$ we have

$$\begin{aligned} t^C \wedge t &= \left(\bigvee\{s \in T : s \wedge t = \emptyset_{\mathcal{T}}\}\right) \wedge t = \\ &= \bigvee\{s \wedge t : s \in T, s \wedge t = \emptyset_{\mathcal{T}}\} = \emptyset_{\mathcal{T}}, \end{aligned} \quad (3.8)$$

so that $I_1(\pi) = \emptyset_{\mathcal{T}}$ immediately follows. The assertion is proved. □

It is perhaps worth noting explicitly that for non-distributive complete lattices the assertion of Theorem 3.1 need not hold. Indeed, taking the non-distributive complete lattice $\mathcal{T}_0 = \langle\{t_0, t_1, t_2, t_3, t_4\}, \leq\rangle$, introduced in Section 2 above, setting $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\pi(\omega_i) = t_i$ for $i = 1, 2, \pi(\omega_3) = \emptyset_{\mathcal{T}}$, we obtain easily that π is a $\mathcal{T}_0$-possibilistic distribution on Ω, but $I_1(\pi) = \mathbf{1}_{\mathcal{T}_0} > \emptyset_{\mathcal{T}_0} = I_2(\pi)$ holds.

Given a complete lattice $\mathcal{T} = \langle T, \leq\rangle$, a $\mathcal{T}$-possibilistic distribution π on Ω is called *orthogonal,* if $\pi(\omega_1) \wedge \pi(\omega_2) = \emptyset_{\mathcal{T}}$ for each $\omega_1, \omega_2, \omega_1 \neq \omega_2$, from Ω. As a matter of fact, if $\mathcal{T}$ is completely distributive and π is orthogonal, then $I_2(\pi) = \emptyset_{\mathcal{T}}$. Indeed, due to the complete distributivity of $\mathcal{T}$ we obtain, for each $\omega \in \Omega$, that

$$\begin{aligned} &(\Pi(\Omega - \{\omega\})) \wedge \pi(\omega) = \\ &= \left(\bigvee_{\omega_1\in\Omega,\omega_1\neq\omega} \pi(\omega_1)\right) \wedge \pi(\omega) = \\ &= \bigvee_{\omega_1\in\Omega,\omega_1\neq\omega} (\pi(\omega_1) \wedge \pi(\omega)) = \emptyset_{\mathcal{T}}, \end{aligned} \quad (3.9)$$

so that

$$I_2(\pi) = \bigvee_{\omega\in\Omega} ((\Pi(\Omega - \{\omega\})) \wedge \pi(\omega)) = \oslash_{\mathcal{T}} \qquad (3.10)$$

follows.

Again, both the conditions imposed on $\mathcal{T}$ and π (complete distributivity and orthogonality) can be proved to be necessary introducing appropriate counter-examples.

Theorem 3.2 Let $\mathcal{T} = \langle T, \leq\rangle$ be a complete lattice, let π be a $\mathcal{T}$-possibilistic distribution on a nonempty set Ω such that $\pi(\omega)$ takes the values $\oslash_{\mathcal{T}}$ or $\mathbf{1}_{\mathcal{T}}$ for each $\omega \in \Omega$, then $I_1(\pi) = \oslash_{\mathcal{T}}$ holds. If there is only one $\omega_0 \in \Omega$ such that $\pi(\omega_0) = \mathbf{1}_{\mathcal{T}}$ holds, then $I_2(\pi) = \oslash_{\mathcal{T}}$, otherwise $I_2(\pi) = \mathbf{1}_{\mathcal{T}}$.

Proof. As $\oslash_{\mathcal{T}}^C = \mathbf{1}_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}}^C = \oslash_{\mathcal{T}}$ holds for each complete lattice $\mathcal{T}$, $(\pi(\omega))^C \wedge \pi(\omega) = \oslash_{\mathcal{T}}$ results for each $\omega \in \Omega$, hence, $I_1(\pi) = \oslash_{\mathcal{T}}$ immediately follows. If $\pi(\omega_0) = \mathbf{1}_{\mathcal{T}}, \pi(\omega) = \oslash_{\mathcal{T}}$ for each $\omega \in \Omega$, $\omega \neq \omega_0$, then $\Pi(\Omega - \{\omega_0\}) = \bigvee_{\omega\in\Omega,\omega\neq\omega_0} \pi(\omega) = \oslash_{\mathcal{T}}$, so that $\Pi(\Omega - \{\omega\}) \wedge \pi(\omega) = \oslash_{\mathcal{T}}$ for every $\omega \in \Omega$ and the equality $I_2(\pi) = \oslash_{\mathcal{T}}$ follows. Otherwise, there are at least two $\omega_1, \omega_2 \in \Omega$ such that $\pi(\omega_1) = \pi(\omega_2) = \mathbf{1}_{\mathcal{T}}$, hence, for each $\omega \in \Omega$, either $\omega_1 \in \Omega - \{\omega\}$ or $\omega_2 \in \Omega - \{\omega\}$ holds, so that $\Pi(\Omega - \{\omega\}) = \mathbf{1}_{\mathcal{T}}$ for every $\omega \in \Omega$. Consequently,

$$\begin{aligned} I_2(\pi) &= \bigvee_{\omega\in\Omega} ((\Pi(\Omega - \{\omega\})) \wedge \pi(\omega)) \geq \\ &\geq ((\Pi(\Omega - \{\omega_1\})) \wedge \pi(\omega_1)) \vee \\ &\vee ((\Pi(\Omega - \{\omega_2\})) \wedge \pi(\omega_2)) = \mathbf{1}_{\mathcal{T}} \end{aligned} \qquad (3.11)$$

follows, so that the assertion is proved. □

4 Entropy Values for Independent Products of Lattice-Values Possibilistic Distributions

A well-known and important property of Shannon entropy reads that for the statistically (stochastically) independent product of probability distributions on discrete spaces the entropy of the resulting probability distribution is given by the sum of the entropy values for the particular distributions. More formally, if for both $i = 1, 2, P_i$ defines a probability distribution on a nonempty discrete space Ω_i, and if $P_{12}(\omega_1, \omega_2) = P_1(\omega_1)P_2(\omega_2)$ for each $\langle\omega_1, \omega_2\rangle \in \Omega_1 \times \Omega_2$, then the equality $I(P_{12}) = I(P_1) + I(P_2)$ holds (for I^* we obtain that $I^*(P_{12}) = 1-((1-I^*(P_1))\,(1-I^*(P_2))$ holds). Let us prove a possibilistic variant of this relation.

Theorem 4.1 Let $\mathcal{T} = \langle T, \leq\rangle$ be a completely distributive complete lattice and, for both $i = 1, 2$, let π_i be a $\mathcal{T}$-possibilistic distribution on a nonempty set Ω_i. Set $\pi_{12}(\omega_1, \omega_2) = \pi_1(\omega_1) \wedge \pi_2(\omega_2)$ for each $\langle\omega_1, \omega_2\rangle \in \Omega_1 \times \Omega_2$. Then π_{12} defines a $\mathcal{T}$-possibilistic distribution on $\Omega_1 \times \Omega_2$ and, for both $j = 1, 2$, the equality $I_j(\pi_{12}) = I_j(\pi_1) \vee I_j(\pi_2)$ holds.

Proof. Applying the assumption of complete distributivity of $\mathcal{T}$, we obtain easily that

$$\begin{aligned} &\bigvee_{\langle\omega_1,\omega_2\rangle} \pi_{12}(\omega_1, \omega_2) = \\ &= \bigvee_{\omega_1\in\Omega_1} \left(\bigvee_{\omega_2\in\Omega_2} (\pi_1(\omega_1) \wedge \pi_2(\omega_2)) \right) = \\ &= \bigvee_{\omega_1\in\Omega_1} \left(\left(\bigvee_{\omega_2\in\Omega_2} \pi_2(\omega_2) \right) \wedge \pi_1(\omega_1) \right) = \\ &= \bigvee_{\omega_1\in\Omega_1} (\mathbf{1}_{\mathcal{T}} \wedge \pi_1(\omega_1)) = \mathbf{1}_{\mathcal{T}}, \end{aligned} \qquad (4.1)$$

so that π_{12} defines a $\mathcal{T}$-possibilistic distribution on $\Omega_1 \times \Omega_2$. A repeated application of the same assumption yields that

$$\begin{aligned} I_2(\pi_{12}) &= \bigvee_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2} [(\Pi_{12}(\Omega_1 \times \Omega_2) - \\ &\quad -\{\langle\omega_1, \omega_2\rangle\}) \wedge \pi_{12}(\omega_1, \omega_2)] = \\ &= \bigvee_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2} \{[\Pi_{12}[((\Omega_1 - \{\omega_1\}) \times \Omega_2) \cup \\ &\quad \cup(\Omega_1 \times (\Omega_2 - \{\omega_2\}))]] \wedge \pi_1(\omega_1) \wedge \pi_2(\omega_2)\} = \\ &= \bigvee_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2} \{[\Pi_1(\Omega_1 - \{\omega_1\}) \vee \Pi_2(\Omega_2 - \{\omega_2\})] \wedge \\ &\quad \wedge\pi_1(\omega_1) \wedge \pi_2(\omega_2\} = \\ &= \bigvee_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2} (\Pi_1(\Omega_1 - \{\omega_1\}) \wedge \pi_1(\omega_1) \wedge \pi_2)) \vee \\ &\vee \bigvee_{\langle\omega_1,\omega_2\rangle\Omega_1\times\Omega_2} (\Pi_2(\Omega_2 - \{\omega_2\}) \wedge \pi_1(\omega_1) \wedge \pi_2(\omega_2)) = \\ &= \bigvee_{\omega_2\in\Omega_2} \left(\left(\bigvee_{\omega_1\in\Omega_1} (\Pi_1(\Omega_1 - \{\omega_1\}) \wedge \pi_1(\omega_1) \right) \wedge \pi_2(\omega_2) \right) \vee \\ &\vee \bigvee_{\omega_1\in\Omega_1} \left(\left(\bigvee_{\omega_2\in\Omega_2} (\Pi_2(\Omega_2 - \{\omega_2\}) \wedge \pi_2(\omega_2) \right) \wedge \pi_1(\omega_1) \right) = \\ &= \bigvee_{\omega_2\in\Omega_2} (I_2(\pi_1) \wedge \pi_2(\omega_2)) \vee \bigvee_{\omega_1\in\Omega_1} (I_2(\pi_2) \wedge \pi_1(\omega_1)) = \\ &= \left(I_2(\pi_1) \wedge \left(\bigvee_{\omega_2\in\Omega_2} \pi_2(\omega_2) \right) \right) \vee \left(I_2(\pi_2) \wedge \right. \end{aligned}$$

$$\wedge\left(\bigvee_{\omega_1\in\Omega_1}\pi_1(\omega_1)\right)\Bigg) =$$

$$= I_2(\pi_1)\vee I_2(\pi_2). \qquad (4.2)$$

For $I_1(\pi_{12})$ the same equality holds trivially, as in this case $I_1(\pi_{12}) = I_1(\pi_1) = I_1(\pi_2) = \oslash_{\mathcal{T}}$ due to Theorem 3.1. The assertion is proved. $\square$

Theorem 4.1 can be easily generalized in this way: let $\mathcal{T}, \Omega_i$ and π_i be as in Theorem 4.1, but this time for $i = 1, 2, \ldots, n$, set $\pi^n(\omega_1, \omega_2, \ldots, \omega_n) = \bigwedge_{i=1}^n \pi_i(\omega_i)$ for each $\langle\omega_1, \omega_2, \ldots, \omega_n\rangle \in \Omega_1\times\Omega_2\times\ldots\times\Omega_n$. Then π^n is a $\mathcal{T}$-possibilistic distribution on $\mathbf{X}_{i=1}^n\Omega_i$ (as can be easily proved by induction) and $I_2(\pi^n) = \bigvee_{i=1}^n I_2(\pi_i)$ holds (for I_1 it is valid trivially). Indeed, for $n = 2$ it is proved in Theorem 4.1, suppose that it is valid for $n-1$. Denoting $\mathbf{X}_{i=1}^{n-1}\Omega_i$ by Ω_1^*, Ω_n by Ω_2^*, π^{n-1} by π_1^* and π_n by π_2^*, and applying Theorem 4.1 to Ω_1^*, π_1^*, Ω_2^* and π_2^*, we obtain that

$$\begin{aligned} & I_2(\pi_n) = I_2(\pi^{n-1}\wedge\pi_n) = I_2(\pi_1^*\wedge\pi_2^*) = \\ = \; & I_2(\pi_{12}^*) = I_2(\pi_1^*)\vee I_2(\pi_2^*) = \\ = \; & \left(\bigvee_{i=1}^{n-1} I_2(\pi_i)\right)\vee I_2(\pi_n) = \bigvee_{i=1}^{n} I_2(\pi_i). \qquad (4.3)\end{aligned}$$

Remark 4.1 As proved in [4], for real-valued possibilistic distributions the same equality is valid also for *infinite* sequences $\langle\Omega_1, \pi_1\rangle, \langle\Omega_2, \pi_2\rangle, \ldots$ of nonempty spaces and possibilistic distributions.

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