A logic determined by commutative residuated lattices

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Abstract

We give an axiom system of a logic CRL which is characterized by the class of commutative residuated lattices. It has as axiomatic extensions well-known logics, UL by Metcalfe and Montanga (ML by Höhle, MTL by Esteva and L.Godo, BL by Hájek). Moreover we show that in any CRL algebra, the following four conditions (PL), $(C_1) + (C_2)$, (E_1^*) and (E_2^*) are equivalent to each other in $\mathcal{CRL}$. This is a generalization of the results proved in [5] and [6].

Keywords: commutative residuated lattice, MTL, BL, UL

1 Introduction

Residuation is a fundamental concept of ordered structures and categories. So far algebraic researches with residuations have done and are doing now for many logics. This is a hot research field in many-valued logic, especially. Many logics are characterized by the class of algebras based on lattices which have residuations. For example, propositional logic, intuitionistic logic, Hájek's BL (basic logic) and Lukasiewizc's MV (many-valued logic) are determined by the class of Boolean algebras, Heyting algebras, BL-algebras and of MV-algebras, respectively. All of these algebras have a common base as algebras, lattices with residuations. Thus, it is very important to develop logics in the view point of residuations. Here we propose an axiom system called CRL which completeness theorem will be proved by the class of *commutative residuated lattices*. The axiom system of the logic CRL has naturally interpreted axioms, that is, all algebraic properties of commutative residuated lattices are reflected directly to axioms of CRL. Thus, it is easy to prove the completeness theorem. Also it gives a uniform treat of many other logics based on commutative residuated lattices, in particular UL by Metcalfe and Montanga ([4]), ML by Höhle ([2]), MTL by Esteva and Godo ([1]), BL by Hájek ([3]), uniform based logic ([7]) and so on.

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2 Logic CRL

We define a logic CRL here, which is determined by the class of all commutative residuated lattices. Thus, we name it CRL. The logic has a following language.

Propositional variables : $p_0, p_1, p_2, \cdots$

Constants : $E, \bot$

Logical symbols : $\wedge, \vee, \rightarrow, \circ$

A formula of CRL is defined as follows:

(1) Every propositional variable is a formula;

(2) Each constant is a formula;

(3) If A and B are formulas, then so are $A \wedge B, A \vee B, A \rightarrow B, A \circ B$.

Let $\Phi_0 = \{p_0, p_1, p_2, \cdots\} \cup \{E, \bot\}$ and Φ be the set of all formulas of CRL. A logical system CRL has the following axioms and rules of inference:

Axioms:

(A1) $A \rightarrow A \vee B$

(A2) $A \vee B \rightarrow B \vee A$

(A3) $A \wedge B \rightarrow A$

(A4) $A \wedge B \rightarrow B \wedge A$

(A5) $A \circ B \rightarrow B \circ A$

(A6) $A \circ E \rightarrow A$

(A7) $A \rightarrow A \circ E$

(A8) $\bot \to A$

(A9) $(A \to B) \to ((B \to C) \to (A \to C))$

Rules of inference

$(R_\vee)$ $\dfrac{A \to C \quad B \to C}{A \vee B \to C}$, $(R_\wedge)$ $\dfrac{A \to B \quad A \to C}{A \to B \wedge C}$

$(R_{I\circ})$ $\dfrac{A \to (B \to C)}{A \circ B \to C}$, $(R_{E\circ})$ $\dfrac{A \circ B \to C}{A \to (B \to C)}$

(MP) $\dfrac{A \quad A \to B}{B}$

A formula A is called *provable* in CRL when there is a finite sequence $A_1, A_2, \cdots, A_n (= A)$ $(n \geq 1)$ of formulas such that, for every i $(1 \leq i \leq n)$,

(1) A_i is an axiom;

(2) A_i is deduced from A_j, A_k $(j, k < i)$ by one of the rules of inference above.

By $\vdash_{CRL} A$, we mean that A is provable in CRL. We note that $A \to (\bot \to \bot)$ is provable for every formula A in CRL. Thus we use a new symbol $\top$ as an abbreviation of the formula $\bot \to \bot$. Thus for every formula A we have $\vdash_{CRL} A \to \top$. It is easy to prove the following.

Proposition 1. *For all $A, B, C \in \Phi$, we have*

(1) $\dfrac{A \to B \quad B \to C}{A \to C}$

(2) $\vdash A \to A$

(3) $\vdash A \circ (A \to B) \to B$

(4) $\dfrac{E \to A}{A}$, $\dfrac{A}{E \to A}$

(5) $\vdash (A \circ B) \circ C \to A \circ (B \circ C)$

(6) $\vdash A \circ (B \circ C) \to (A \circ B) \circ C$

(7) $\dfrac{A \to B}{A \wedge C \to B \wedge C}$

(8) $\dfrac{A \to B}{A \vee C \to B \vee C}$

(9) $\dfrac{A \to B}{A \circ C \to B \circ C}$

(10) $\dfrac{A \to B}{(B \to C) \to (A \to C)}$

(11) $\dfrac{A \to B}{(C \to A) \to (C \to B)}$

3 CRL algebra

We define an algebraic semantics of the logic CRL, which gives a completeness theorem of CRL. By CRL algebra $(X, \wedge, \vee, \cdot, \to, 0, e, 1)$, we mean the following algebra

(1) $(X, \wedge, \vee, 0, 1)$ is a bounded lattice

(2) $(X, \cdot, e)$ is a commutative monoid with unit element e

(3) For all $x, y, z \in X$, $x \cdot y \leq z$ if and only if $x \leq y \to z$.

As to properties of CRL algebras, we have the following.

Proposition 2. *Let X be a CRL algebra. For all $x, y, z \in X$, we have*

(1) $x \leq y \iff e \leq x \to y$

(2) $x \cdot (x \to y) \leq y$

(3) $x \leq y \implies x \cdot z \leq y \cdot z,\ z \to x \leq z \to y,\ y \to z \leq x \to z$

(4) $e \to x = x$

(5) $x \to 1 = 1$

(6) $1 \to x \leq x$

(7) $e \leq 0 \to x$

(8) $1 \cdot 1 = 1$

(9) $(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z)$

Let X be a CRL algebra. A map $v : \Phi_0 \to X$ is called a *valuation* on X and it can be extended uniquely to the set Φ of all formulas as follows:

(1) $v(A \wedge B) = v(A) \wedge v(B)$

(2) $v(A \vee B) = v(A) \vee v(B)$

(3) $v(A \to B) = v(A) \to v(B)$

(4) $v(A \circ B) = v(A) \cdot v(B)$

We denote the extended valuation v by the same symbol v. It is easy to show that $v(\bot) = 0, v(E) = e$ and $v(\top) = 1$. The following is easy to prove.

Lemma 1 (Soundness Theorem)**.** *For every formula A, if $\vdash_{CRL} A$ then $v(A) \geq e$ for any valuation v on any CRL algebra X.*

We use the well-known method called *Lindenbaum-Tarski* algebra to prove the converse direction (Completeness Theorem) of the above. At first we define a relation $\equiv$ on the set Φ of formulas of CRL : For $A, B \in \Phi$,

$$A \equiv B \iff \vdash_{CRL} A \to B \text{ and } \vdash_{CRL} B \to A$$

As to the relation $\equiv$, we see directly from the proposition 1 that

Lemma 2. *$\equiv$ is a congruence on Φ, that is, it is an equivalence relation and satisfies the substitution property : If $A \equiv B$ and $C \equiv D$, then*

$$A \wedge C \equiv B \wedge D,\ A \vee C \equiv B \vee D,$$
$$A \to C \equiv B \to D,\ A \circ C \equiv B \circ D$$

Since $\equiv$ is the congruence, we can define operations on $\Phi/\equiv$: For $A, B \in \Phi$, we define

$$[A] \sqcap [B] = [A \wedge B],$$
$$[A] \sqcup [B] = [A \vee B],$$
$$[A] \circ [B] = [A \circ B],$$
$$[A] \to [B] = [A \to B],$$
$$\mathbf{0} = [\bot],\ \mathbf{e} = [E],\ \mathbf{1} = [\top].$$

Lemma 3. *$(\Phi/\equiv, \sqcap, \sqcup, \circ, \to, \mathbf{0}, \mathbf{e}, \mathbf{1})$ is a CRL algebra.*

We note that if $\vdash_{CRL} A \to B$ then $[A] \le [B]$, that is, $\mathbf{e} \le [A] \to [B]$ in the CRL algebra $\Phi/\equiv$.

We define a valuation $V^* : \Phi_0 \to \Phi/\equiv$ as $V^*(p) = [p]$, then we have $V^*(A) = [A]$ for every formula A of CRL.

Lemma 4. *For any formula $A \in \Phi$,*

$$\vdash_{CRL} A \iff V^*(A) \ge \mathbf{e} \text{ in } \Phi/\equiv$$

Proof. If $\vdash A$, since $\vdash A \circ E \to A$, then we have $\vdash A \to (E \to A)$. It follows from assumption that $\vdash E \to A$. This means that $\mathbf{e} \le [A]$.

Conversely, $\mathbf{e} \le [A]$ implies $\vdash E \to A$. It follows from proposition 1 that $\vdash A$. □

From the above, we can prove the next theorem.

Theorem 1 (Completeness Theorem). *Let $A \in \Phi$. Then we have*

> *$\vdash_{CRL} A$ if and only if $v(A) \ge e$ for every valuation $v : \Phi_0 \to X$ on any CRL algebra X.*

Proof. Suppose that $v(A) \ge e$ for every valuation $v : \Phi_0 \to X$ on any CRL algebra X. Thus, in particular, we have $V^*(A) \ge \mathbf{e}$ for the valuation V^* on the CRL algebra $\Phi/\equiv$. From the lemma above it follows that $\vdash A$. □

Next we consider relations between our logic and other familiar logics, MTL, ML, UL and so on. The following is the axiom system of MTL according to [4]:

Axioms:

(MTL1) $(A \to B) \to ((B \to C) \to (A \to C))$

(MTL2) $A \circ B \to A$

(MTL3) $A \circ B \to B \circ A$

(MTL4) $A \wedge B \to A$

(MTL5) $A \wedge B \to B \wedge A$

(MTL6) $A \circ (A \to B) \to A \wedge B$

(MTL7a) $(A \to (B \to C)) \to (A \circ B \to C)$

(MTL7b) $(A \circ B \to C) \to (A \to (B \to C))$

(MTL8) $((A \to B) \to C) \to (((B \to A) \to C) \to C)$

(MTL9) $\bot \to A$

Rules of inference

(MP) $\dfrac{A \quad A \to B}{B}$

It is easy to prove that our logic CRL is equivalent to the system MTL without (I) $A \to (B \to A)$ and $(PL1)$ $(A \to B) \vee (B \to A)$. That is, MTL is an extension of our logic CRL with two axioms (I) and $(PL1)$. More accurately, we take $\mathcal{CRL} + (I) + (PL1)$ as a logic which has extra axioms (I) $A \to (B \to A)$ and (PL1) $(A \to B) \vee (B \to A)$ besides those of CRL. Then we can show without difficulty that MTL is equivalent to the logic CRL + (I) + (PL1), that is,

$$\vdash_{MTL} A \text{ if and only if } \vdash_{CRL+(I)+(PL1)} A.$$

4 Extension of CRL logic

By adding some extra axioms to CRL according to [5] and [6], we can consider interesting subvarieties of the variety of commutative residuated lattices. We list axioms to be added to CRL.

(I) $A \to (B \to A)$

$(PL1)$ $(A \to B) \vee (B \to A)$

$(PL2)$ $(E \wedge (A \to B)) \vee (E \wedge (B \to A))$

(C_1) $E \to (A \to B) \vee (B \to A)$

(C_2) $E \wedge (A \vee B) \to (E \wedge A) \vee (E \wedge B)$

(E_1) $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$

(E_2) $(A \to B \vee C) \to (A \to B) \vee (A \to C)$

(PL) $E \to (E \wedge (A \to B)) \vee (E \wedge (B \to A))$

(E_1^*) $(A \wedge B \to C) \wedge E \to ((A \to C) \wedge E) \vee ((B \to C) \wedge E)$

(E_2^*) $(A \to B \vee C) \wedge E \to ((A \to B) \wedge E) \vee ((A \to C) \wedge E)$

These axioms are correspond to the following conditions, respectively.

(I) e is the greatest element, that is, $e = 1$.

$(PL1)$ $(a \to b) \vee (b \to a) = 1$

$(PL2)$ $(e \wedge (a \to b)) \vee (e \wedge (b \to a)) \geq e$

(C_1) $e \leq (a \to b) \vee (b \to a)$

(C_2) $e \wedge (a \vee b) \leq (e \wedge a) \vee (e \wedge b)$

(E_1) $(a \wedge b \to c) \leq (a \to c) \vee (b \to c)$

(E_2) $(a \to b \vee c) \leq (a \to b) \vee (a \to c)$

(PL) $e \leq (e \wedge (a \to b)) \vee (e \wedge (b \to a))$

(E_1^*) $(a \wedge b \to c) \wedge e \leq ((a \to c) \wedge e) \vee ((b \to c) \wedge e)$

(E_2^*) $(a \to b \vee c) \wedge e \leq ((a \to b) \wedge e) \vee ((a \to c) \wedge e)$

As the similar argument above, we see that the *monoidal logic* ML ([2]) is the CRL logic with the extra axiom (I), the *monoidal t-norm logic* MTL ([1]) is the CRL logic with (I) and $(PL1)$ and the *uninorm logic* UL [4] is one with $(PL2)$. We represent such situation by the following informal form.

$$ML = CRL + (I)$$
$$MTL = CRL + (I) + (PL1)$$
$$UL = CRL + (PL2)$$

As to these axioms we have the following fundamental result.

Proposition 3. *For all formulas A, B, we have $\vdash_{CRL} A \to (B \to A)$ if and only if $\vdash_{CRL} \top \to E$.*

From the above we see that $\vdash_{ML} A$ if and only if A is valid on the class of all CRL algebras with the condition that e is the largest element, that is, $e = 1$ in the lattice. It is similar to the other logics. Thus we have completeness theorems of these other logics.

Theorem 2. *Let $A \in \Phi$ and $\alpha, \cdots, \beta$ be some of conditions $\{(I), (PL1), \cdots, (E_1^*), (E_2^*)\}$ described above. Then we have*

$\vdash_{CRL+\{\alpha,\cdots,\beta\}} A$ if and only if $v(A) \geq e$ for every valuation $v : \Phi_0 \to X$ on any CRL algebra X with the corresponding conditions $\alpha, \cdots, \beta$.

For the case of MTL (or UL) algebra, it is known ([5]) that all subdirectly irreducible MTL (UL)-algebras are totally ordered. Thus it follows that the logic MTL (UL) is characterized by the totally ordered MTL (UL)-algebras, that is,

$\vdash_{MTL} A$ $(\vdash_{UL} A)$ if and only if $v(A) = 1$ $(v(A) \geq e)$ for every valuation $v : \Phi_0 \to X$ on any *totally ordered* MTL (UL) algebras X.

Moreover the class $\mathcal{MTL}$ of all MTL algebras is the least distributive variety containing the class $\mathcal{WL}$ of all CRL algebras with (C_1) $(x \to y) \vee (y \to x) \geq e$. Indeed, let $\mathcal{V}$ be the class of all distributive CRL algebras containing $\mathcal{WL}$. Since any algebra A in $\mathcal{MTL}$ is distributive, it satisfies the condition (C_2) $e \wedge (a \vee b) = (e \wedge a) \vee (e \wedge b)$. Thus A satisfies (C_1) and (C_2). It follows from the result in [5] that $\mathcal{MTL} \subseteq \mathcal{V}$.

As to the other conditions above, it is proved that

- Three conditions (C_1), (E_1) and (E_2) are equivalent to each other in $\mathcal{CRL}$ with (I) ([6]) ;
- Three conditions (C_1), (E_1) and (E_2) are equivalent to each other in $\mathcal{CRL}$ with (C_2) ([5]).

We now show that

Theorem 3. *The following four conditions (PL), (C_1) + (C_2), (E_1^*) and (E_2^*) are equivalent to each other in $\mathcal{CRL}$.*

Proof. We only show that (PL) implies (E_1^*). Since the class of all CRL algebras with (PL) is the variety whose subdirectly irreducible members are just totally ordered CRL algebras ([5]), it is enough to show that any totally ordered CRL algebra satisfies the condition (E_1^*). Simple calculation yields to the desired result. □

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