The logic of perfect MV-algebras

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Abstract

In this paper we summarize results on perfect MV-algebras and we show that the class of first order Łukasiewicz formulas valid in all interpretations over perfect MV-chains is axiomatizable.

Keywords: MV-algebras, perfect algebra, first order Łukasiewicz logic.

1 Introduction

The class of MV-algebras arises as algebraic counterpart of the infinite valued Łukasiewciz sentential calculus, as Boolean algebras did with respect to the classical propositional logic. Due to the non-idempotency of the MV-algebraic conjunction, unlike Boolean algebras, MV-algebras can be non-archimedean and can contains elements x such that $x \odot \ldots \odot x$ (n times) is always greater than zero, for any $n > 0$ (here $\odot$ denotes the Łukasiewicz conjunction). In general, there are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. Non-zero elements from the radical of A are called *infinitesimals*. Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical. Hence perfect MV-algebras can be seen as extreme examples of non-archimedean MV-algebras.

An important example of a perfect MV-algebra is the subalgebra S of the Lindenbaum algebra L of first order Łukasiewicz logic generated by the classes of formulas which are valid when interpreted in $[0, 1]$ but non-provable. Hence perfect MV-algebras are directly connected with the very important phenomenon of incompleteness in Łukasiewicz first order logic (see [10], [2]).

As it is well known, MV-algebras form a category which is equivalent to the category of abelian lattice ordered groups (ℓ-groups, for short) with strong unit [9]. This makes the interest in MV-algebras relevant outside the realm of logic. Let us denote by Γ the functor implementing this equivalence. In particular each perfect MV-algebra is associated with an abelian ℓ-group with a strong unit. But, more has been proved. Namely the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups, see ([7], Theorem 3.5, p.420). Let us denote by $\mathcal{D}$ the functor implementing this equivalence. Hence $\mathcal{D}$ maps functorially each perfect MV-algebra to an abelian ℓ-group and vice versa, without the help of a strong unit. Here a curious remark has to be made. Indeed the Γ functor maps a non-equational class, the category of abelian ℓ-groups with strong unit, to an equational class, the variety of MV-algebras. On the other hand, the functor $\mathcal{D}$ maps an equational class, the category of abelian ℓ-groups, to a non-equational class, the category of perfect MV-algebras. However, as a consequence of using the functor $\mathcal{D}$, a surprising result was proved showing the equivalence between the category of perfect MV-algebras with a distinguished generator of the radical, whose morphisms preserve the distinguished element, and the category of all MV-algebras, cite4.

Perfect MV-algebras do not form a variety and contains non-simple subdirectly irreducible MV-algebras. It is worth stressing that the variety generated by all perfect MV-algebras is also generated by a single MV-chain, actually the MV-algebra C, defined by Chang in [5]. The MV-algebra C is therefore a prototypical perfect MV-algebra, and we shall see it plays a crucial role in the theory of perfect MV-algebras, as well as in the variety it generates.

In this paper we recount basic properties of perfect MV-algebras and we adapt results from [2] for establishing that the class of first order Łukasiewicz formulas valid on all interpretations over perfect MV-chains is axiomatizable.

2 Perfect MV-algebras

A structure $A = (A, 0, 1, \neg, \odot, \oplus)$ is an MV-algebra iff A satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $\neg 0 = 1$;
6. $\neg 1 = 0$;
7. $x \odot y = \neg(\neg x \oplus \neg y)$;
8. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

On A two new operations $\vee$ and $\wedge$ are defined as follows: $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \odot y) \odot y$. The structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. We shall write $x \leq y$ iff $x \wedge y = x$. We say that an MV-algebra A is an MV-chain when, as a lattice, A is linearly ordered. Boolean algebras are just the MV-algebras obeying the additional equation $x \odot x = x$. For any MV-algebra A we denote by $B(A) = \{x \in A \mid x \odot x = x\}$ the biggest Boolean algebra contained in A. We write nx instead of $x \oplus ... \oplus x$ (n-times) and x^n instead of $x \odot ... \odot x$ (n-times). The least integer for which $nx = 1$ is called the *order* of x. When such an integer exists, we denote it by $ord(x)$ and say that x has *finite order*, otherwise we say that x has *infinite order* and write $ord(x) = \infty$.

Example. The unit interval of real numbers $[0, 1]$ with operations defined by $x \oplus y = \min\{1, x + y\}$, $x \odot y = \max\{0, x + y - 1\}$, and $\neg x = 1 - x$ is an MV-algebra. We shall refer to this MV-algebra as $[0, 1]$.

An *ideal* of an MV-algebra A is a non-empty subset I of A which is closed under $\oplus$ and such that if $x \leq y$ and $y \in I$ then $x \in I$. A *prime* ideal P of A is an ideal of A such that $x \wedge y$ implies $x \in P$ or $y \in P$. An ideal M of A is called *maximal* if $M \subseteq I$ implies $I = A$ or $I = M$, where I an ideal of A. Let M be a maximal ideal of A, then we say that M is *supermaximal* if $A/M \cong \{0, 1\}$. The set of all prime ideals of A shall be denoted by $Spec(A)$. For each element x of an MV-algebra A the set

$$id(x) = \{y \in A \mid y \leq nx, \text{ for some } n > 0\}$$

is the ideal of A generated by x. Each proper ideal is contained in a maximal ideal.

As MV-algebras form an equational class, the notions of MV-isomorphism, quotient, subalgebra, product, etc., are just the particular cases of the corresponding universal algebraic notions.

The intersection of all maximal ideals, the *radical* of A, will be denoted by $Rad(A)$.

An MV-algebra A such that $Rad(A) = 0$ is called *semisimple*. An MV-algebra A is called *simple* if and only if A is non trivial and $\{0\}$ is its only proper ideal. Every simple MV-algebra is isomorphic to a subalgebra of $[0, 1]$, (see, e.g., [6], Theorem 3.5.1, p. 70). Every non-zero element of a non trivial MV-algebra A has finite order if and only if A is simple.

If for every element x of the MV-algebra A there is an integer n such that nx is idempotent then A will be called *hyperarchimedean*. For all unexplained MV-algebraic notions we refer the reader to [6].

Definition 1. An MV-algebra A is *local* if A has a unique maximal ideal. The class of all local MV-algebras will be denoted by **Local**.

It is also well known that for each $x \in A$, x is a member of a proper ideal, hence a maximal ideal, if and only if the order of x is ∞.

It turns out that an MV-algebra A is local if and only if for every $x \in A$, $ord(x) < \infty$ or $ord(\neg x) < \infty$.

Definition 2. An MV-algebra A is called *perfect* if for every nonzero element $x \in A$ $ord(x) = \infty$ if and only if $ord(\neg x) < \infty$. The class of all perfect MV-algebras will be denoted by **Perfect**.

It is clear that the class of all MV-algebras is a variety, here denoted by MV. For any subclass K of elements from MV, $V(K)$ shall denote the subvariety of MV generated by K. If K has just one element A then we also write $V(A)$ for $V(K)$.

Definition 3. Chang's MV-algebra is defined on the set

$$C = \{0, c, \ldots, nc, \ldots, 1 - nc, \ldots, 1 - c, 1\}$$

by the following operations (consider $0 = 0c$):
$x \oplus y =$

- $(m + n)c$ if $x = nc$ and $y = mc$
- $1 - (m - n)c$ if $x = 1 - nc$ and $y = mc$ and $0 < n < m$
- $1 - (n - m)c$ if $x = nc$ and $y = 1 - mc$ and $0 < m < n$
- 1 otherwise;

$$\neg x = \begin{cases} 1 - nc & \text{if } x = nc \\ nc & \text{if } x = 1 - nc \end{cases}$$

C is a linearly ordered MV-algebra, $ord(nc) = \infty$ and $ord(1 - nc) < \infty$ for every n. So C is a perfect MV-algebra.

Definition 4. A proper ideal P of an MV-algebra A is called *perfect* if and only if for every $a \in A$, $a^n \in P$ for some $n \in \omega$ if and only if $(\neg a)^m \notin P$ for all $m \in \omega$.

P is a perfect ideal if and only if A/P is a perfect MV-algebra.

Proposition 5. *The following hold:*

1. *The only finite perfect MV-algebra is* $\{0, 1\}$*;*
2. *Every nonzero element in a perfect MV-algebra* $A \neq B(A)$ *generates a subalgebra isomorphic to the Chang MV-algebra* C*.*
3. *Linearly ordered MV-algebras in* $V(C)$ *are all perfect MV-chains;*
4. $V(\mathbf{Perfect}) = V(C)$*,*
5. $\mathbf{Perfect} = V(C) \cap \mathbf{Local}$*;*
6. $A \in V(C)$ *iff for every* $x \in A$*,* $2x^2 = (2x)^2$*;*
7. A *is perfect iff* $A = alg(Rad(A))$*;*
8. A *is perfect iff* $A = Rad(A) \cup \neg(Rad(A))$*; further* $x \in Rad(A)$ *iff* $ord(x) = \infty$*;*
9. **Perfect** *is closed under homomorphic images and subalgebras;*
10. A *is perfect iff any proper ideal of* A *is perfect iff* $\{0\}$ *is a perfect ideal.*

Proof. Properties (1) and (2) are immediate consequences of order of elements in a perfect MV-algebra.

The proof of properties $(3), (4), (5), (6)$ and of $(7), (8), (9), (10)$ can be found in [7] and in [4] respectively. ☐

Now we are going to show that the class of perfect MV-algebras is first order definable.

Consider the following well formed formulas in the first order language of MV-algebras containing the equality relation as predicate symbol, operations of MV-algebras as functional symbols and 0 and 1 as constant symbols. Further denote by &, OR and $\Rightarrow$ the classical propositional connectives. Let σ be the wff $(\forall x)(x^2 \oplus x^2 = (x \oplus x)^2)$ and τ be the wff $(\forall x)(x^2 = x \Rightarrow (x = 0 \text{ OR } x = 1))$. Then,

Proposition 6. *Let A be an MV-algebra then the following are equivalent:*

(i) A is perfect;

(ii) A satisfies $\sigma \,\&\, \tau$.

Proof. $(i) \Rightarrow (ii)$. Let A be a perfect MV-algebra. Then by Proposition 5(4), $A \in V(\mathbf{C})$ hence A satisfies the equation σ satisfied by C. Further, by Proposition 5(2), A cannot contain idempotent elements different from 0 and 1.

$(ii) \Rightarrow (i)$. Assume A to be an MV-algebra satisfying the formula $\sigma \,\&\, \tau$, so $x^2 = x$ implies $x = 0$ or $x = 1$. Hence $B(A) = \{0, 1\}$ and $A \in V(C)$. Hence, by Chang Theorem (see [6], Theorem 1.3.3, p.20), Theorem 5.1 in ([7] p.424) and Corollary 5.2 in ([7] p.425) (cfr Proposition 5(3)):

$$A \hookrightarrow \prod_{P \in Spec(A)} A/P$$

where, for every $P \in Spec(A)$, A/P is a perfect MV-chain. Assume A is not perfect. Then there is $z \in A$ such that $z \notin Rad(A) \cup \neg(Rad(A))$. Therefore, there are $P, Q \in Spec(A)$ such that

$$z/P \in Rad(A/P) \qquad \text{and} \qquad z/Q \in \neg Rad(A/Q).$$

So we get $2(z^2/P) = 0/P$ and $2(z^2/Q) = 1/Q$, that is $2z^2 \in B(A) \setminus \{0, 1\}$, contradicting $B(A) = \{0, 1\}$. ☐

3 Łukasiewicz logic

In this section we consider first-order Łukasiewicz logic. We shall use definitions and axioms from [2] and [3], but apart from some technicalities, such definitions are analogous to those for example in [8].

Definition 7. Let $\mathcal{L}$ be a language containing symbols of variables $v_0, v_1, \ldots$, logical symbols $\rightarrow, \neg$; predicate symbols $R_0, R_1, \ldots$; a quantifier symbol $\exists$; improper symbols (,); and a function $d : \mathbb{N} \rightarrow \mathbb{N}$, $\mathbb{N} = \{0, 1, 2, ...\}$.

The set of well-formed formulas of $\mathcal{L}$, WFF, is defined as usual, as follows: atomic formulas, $R_n(v_{i_1}, v_{i_2}, ..., v_{i_{d(n)}})$ are in WFF. If $\alpha, \beta \in WFF$ so are $(\alpha \rightarrow \beta)$ and $\neg\alpha$. If $\alpha \in WFF$ and x is a variable then $(\exists x)\alpha$ is in WFF.

Definition 8. Let A be an MV-algebra and X a nonempty set. An $\{A, X\} - model$ is a system $\langle A, X, (F_n)_{n \in \mathbb{N}} \rangle$ such that for each $n \in \mathbb{N}$ there is a function $F_n : X^{d(n)} \rightarrow A$.

An $\{A, X\} - model$ is *linear* if A is an MV-chain, is *canonical* if $A \subseteq [0, 1]$. Given an $\{A, X\} - model$ $\langle A, X, (F_n)_{n \in \mathbb{N}} \rangle$, an *assignment* is a function $f : Var \rightarrow X$, with $Var = \{v_0, v_1, \ldots\}$, i.e., Var is the set of variables of $\mathcal{L}$. If f is an assignment, $v \in Var$, $x \in X$, then f_{vx} is the assignment

$$f_{vx}(v_i) = \begin{cases} f(v_i), & \text{if } v_i \neq v \\ x, & \text{if } v_i = v. \end{cases}$$

If $S \subseteq A$ we define

$$\sum S = \begin{cases} \text{least upper bound of } S \text{ in } A \text{ (if it exists)} \\ A, \quad \text{otherwise.} \end{cases}$$

Given $\mathfrak{M} = \langle A, X, (F_n)_{n\in\mathbb{N}} \rangle$ we assign values to each $\alpha \in WFF$. We define, therefore a function

$$Val(\alpha, \mathfrak{M}, f) : WFF \to A \cup \{A\},$$

where f is an $\mathfrak{M}$-assignment inductively defined by the following conditions:

(1) $Val(R_n(v_{i_1}, v_{i_2}, ..., v_{i_{d(n)}}), \mathfrak{M}, f) = F_n(f(v_{i_1}), f(v_{i_2}), ..., f(v_{i_{d(n)}}))$;

(2) Assuming Val defined for $\alpha \in WFF$, then

$$Val(\neg\alpha, \mathfrak{M}, f) =$$

$$= \begin{cases} \neg Val(\alpha, \mathfrak{M}, f) & \text{if } Val(\alpha, \mathfrak{M}, f) \in A \\ A, & \text{otherwise;} \end{cases}$$

(3) Assuming Val defined for $\alpha, \beta \in WFF$, then

$$Val(\alpha \to \beta, \mathfrak{M}, f) =$$

$$= \begin{cases} \neg Val(\alpha, \mathfrak{M}, f) \oplus Val(\beta, \mathfrak{M}, f), \\ \quad \text{provided } Val(\alpha, \mathfrak{M}, f), Val(\beta, \mathfrak{M}, f) \in A \\ A, \quad \text{otherwise;} \end{cases}$$

(4) $Val((\exists v)\alpha, \mathfrak{M}, f) = \sum_{x\in X} Val(\alpha, \mathfrak{M}, f_{vx})$.

Call an $\mathfrak{M}$-assignment an *interpretation* if $Val(\alpha, \mathfrak{M}, f) \in A$ for all $\alpha \in WFF$.

Axioms of first order Łukasiewicz logic (see for example [2] or [8]), includes all axioms for Łukasiewicz propositional logic. We can hence define the Lindenbaum MV-algebra L of first order Łukasiewicz logic as the quotient of the MV-algebra of all formulas with respect to the congruence given by provable equivalence of formulas. An element from L shall be denoted by $[\alpha]$, with α a formula from $\mathcal{L}$.

Let $\beta_0, \beta_1, ...$ be an enumeration of all $\beta \in WFF$ of the form $\beta_n = (\exists v_i)\alpha_n$. For each $n, m \geq 0$, let $\beta(n, m)$ be defined from α_n by the following steps:

1) let v_j be the first variable not occurring in α_n;

2) replace all bounded occurrences of v_m in α_n by v_j;

3) replace all free occurrences of v_i in α_n by v_m.

An ideal I in L is said to *preserve sums* if for any n, if $[\beta(n, m)] \in I$ for all m, then $\sum_{m\in\mathbb{N}}[\beta(n, m)] \in I$.

The following results are from [2] and [3].

Theorem 9. *Let I be an ideal of L preserving sums. Then the map $\alpha \in Form \mapsto [\alpha]/I \in L/I$ is an interpretation over L/I.*

We know from [2] that the class of first order formulas valid in all interpretations over all linearly ordered MV-algebras is axiomatizable, while the class of formulas valid in interpretations over $[0, 1]$ is not (see [10]). We shall study now the class of first order formulas valid in all interpretation over perfect linearly ordered MV-algebras.

Add to $\mathbf{L}$ the axiom schema, $2\alpha^2 \leftrightarrow (2\alpha)^2$ and denote the resulting system by $\mathbf{L_p}$. Then we have,

Proposition 10. *A wff α is provable in $\mathbf{L_p}$, iff α is valid on all perfect MV-chains.*

Proof. The proof is based on the fact that for any formula β with $ord([\beta]) = \infty$ there exists prime ideal P of L containing $[\beta]$ and preserving sums, hence by Theorem 9 we can define an assignment of formulas into L/P mapping $[\beta]$ into 0. Since the axioms of $\mathbf{L_p}$ include the axioms of $\mathbf{L}$, the same is true for the Lindenbaum algebra L_p of $\mathbf{L_p}$. But the latter algebra is now in the variety $V(C)$ generated by Chang's algebra C (Proposition 5(6)). Consequently each linearly ordered subalgebra of L_p is perfect (Proposition 5(3)). From this it follows that if α is valid on all perfect MV-chains and not provable in $\mathbf{L_p}$, then in L_p, $[\alpha] \neq 1$, hence $\neg[\alpha] \neq 0$. Thus for some prime ideal P we have $\neg[\alpha] \notin P$ and so in L_p/P we have $[\alpha]/P \neq 1$. But L_p/P is a perfect linearly ordered MV-algebra, hence α is not equal to one in a perfect MV-chain, in contradiction with our hypothesis. ☐

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