Inverse arithmetic operators for fuzzy intervals

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Abstract

Fuzzy arithmetic is a powerful tool in many engineering problems such as decision making, control theory, fuzzy systems and approximate reasoning. However, it is well known that the practical use of standard fuzzy arithmetic operators gives results more imprecise than necessary or in some cases, even incorrect. This problem is due to the overestimation effect induced by computing fuzzy arithmetic operations. In this paper a modified implementation for fuzzy unimodal interval arithemtics is defined where new subtraction and division operators are proposed. These new operators are exactly the inverse of the addition and multiplication operators. The effectiveness of the proposed methodology is illustrated by simulation examples.

Keywords: Fuzzy arithmetic, Inverse operators

1 Introduction

Nowadays, fuzzy numbers and fuzzy values are widely used in engineering applications because of their suitability for representing uncertain information. In this context, fuzzy arithmetics using fuzzy numbers and fuzzy intervals [5][12] is a frequently encountered problem, especially in decision making, control theory, fuzzy systems and approximate reasoning problems.

It is known that real numbers arithmetic operations are computed using unique rules that are independent of what is represented by the numbers involved. That is, the result of each particular arithmetic operation on real numbers depends only on the numbers involved and not on the entities represented by the numbers [19][21]. As is well known, the validity of this simple principle is also tacitly assumed in fuzzy arithmetic operations [4][16][17][24]. Indeed, the usual arithmetic operations on real numbers can be extended to the ones defined on fuzzy numbers by means of Zadeh's extension principle [26][27]. In this context, direct implementation of this principle in fuzzy arithmetics is computationally expensive due to the requirement of solving a nonlinear programming problem [2][18]. To overcome this deficiency, many researchers have investigated this problem by observing the fuzzy numbers as a collection of α-levels [1][9][10][20]. In this case, fuzzy arithmetics are performed using conventional interval arithmetics according to the α-cut representation.

In this framework, it has been proved [6][7] that the conventional interval arithmetic operations $\{\oplus, \ominus, \otimes, \oslash\}$ defined for intervals [13][14] can be directly extended to fuzzy intervals using the left and right profiles instead of the lower and upper limits of the intervals. That is the approach used in this paper for computing fuzzy arithmetic operations where unimodal fuzzy intervals are considered.

It is well known that standard interval arithmetic operations give a guaranteed enclosure of the solution, which is usually too large or more imprecise (the so-called "dependence effect"). The origin of the problem is the memoryless nature of interval arithmetic operations [13][14]. So, the independence property assumed in real number arithmetics holds not true for computing interval arithmetics as well as fuzzy interval arithmetics [21][22]. As mentioned by Zhou in [25] and stated by Piegat in [19]: «*the standard fuzzy arithmetic does not take into account all the information available, and the obtained results are more imprecise than necessary or in some cases, even incorrect*». Indeed, as discussed in [21], the fuzzy arithmetic operations tackle into account only the information contained in the operands and completely ignore the additional information that may emanate from the meaning of these operands. It follows that fuzzy arithmetic operations may produce some counterintuitive results. For example, when considering a fuzzy interval A, according to standard fuzzy arithmetics we have $A \ominus A \neq 0$ and $A \oslash A \neq 1$. However, in engineering applications, it can be desirable to have crisp values for $A \ominus A$, and $A \oslash A$, i.e. the crisp values 0 and 1 respectively.

Moreover, it can be stated that the X solution of the fuzzy linear equation $B \oplus X = A$ is not, as we would expect, the value $X = A \ominus B$, and the solution of the fuzzy equation $B \otimes X = A$ is not the fuzzy interval $X = A \oslash B$. Consequently, the operations of addition and subtraction (multiplication and division) of fuzzy intervals are not inverse operations.

In order to overcome these problems, the authors [21][19] propose to tackle into account all constraints which represent additional information that are imposed on variable values in the considered problem. If these constraints (equality constraints, inequality constraints, ...) are respected, fuzzy arithmetics gives exact operations. However, if they are ignored, the obtained results are less precise than necessary. An elegant study of this problem has been developed in [19][21].

In this paper another way is explored to determine exact fuzzy arithmetics operations. The proposed methodology is not based on additional information constraints on the variables but on the characterization of the volume and/or the quantity of uncertainties contained in the handled variables, for which exact fuzzy arithmetics can be obtained.

When considering unimodal fuzzy intervals the originality of this work lies in the answers given to the following questions:

- *Is it possible to develop arithmetic operators for fuzzy intervals so that addition and multiplication are exactly inverse operations of subtraction and division?*
- *Is it possible to determine conditions for guaranteeing the existence of these exact operators?*

This paper is structured in the following way. In section 2, the concepts of fuzzy intervals and fuzzy profiles are introduced. Section 3 is devoted to the presentation of conventional fuzzy arithmetic operations according to the fuzzy profiles principle. In section 4, new fuzzy arithmetic operators are proposed with their existence conditions. Concluding remarks are finally given in Section 5.

2 Fuzzy Intervals and Fuzzy Profiles

An interval a can be considered as a set of elements to which a rectangular membership function $\mu_a(x)$ is associated (see figure 1). It can thus be viewed as a special fuzzy number whose membership function takes the value 1 over the interval and 0 anywhere else. Therefore, interval analysis can be considered as a subset of fuzzy set theory [15][23].

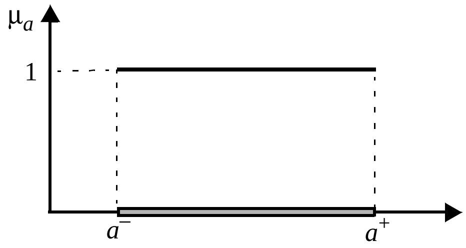


Figure 1: Interval Representation

The interval representation supposes that all possible values of the interval a belong to it with the same membership degree. When using a unimodal fuzzy interval representation, a possibility distribution, represented by a membership function, is associated with the fuzzy interval A whose support is the interval of all possible values and whose kernel value is the one and only best value (see figure 2).

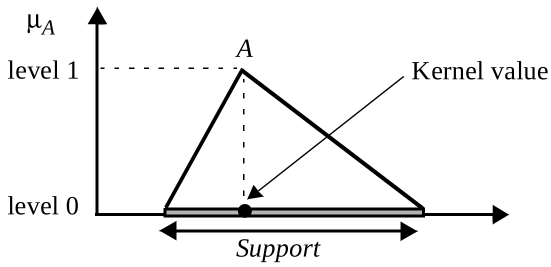


Figure 2: Fuzzy Interval Representation

In order to extend interval arithmetics to fuzzy intervals, one has to consider two dimensions. The first one (horizontal dimension) is similar to that used in interval representation, that is the real line $\Re$. The second one (vertical dimension) is related to the handling of the membership degrees and thus restricted to the interval [0, 1]. In this context, two kinds of information are required for completely defining fuzzy intervals and thus extending the conventional interval operations to the latter. Both pieces of information, called support and kernel value, are defined on the horizontal dimension, but are associated to two different levels (level 0 and level 1) on the vertical dimension (see figure 2).

Generally, an interval valued number is defined by the set of elements lying between its lower and upper limits as $a = \{x \mid a^- \leq x \leq a^+, x \in \Re\}$. Given an interval a, its midpoint $M(a)$ and its radius $R(a)$ are defined by:

$$M(a) = (a^- + a^+) / 2 \quad \text{and} \quad R(a) = (a^+ - a^-) / 2 \quad (1)$$

Let us now consider a unimodal fuzzy interval A [5] whose membership function is denoted μ_A and whose above mentioned features, i.e. support and kernel value, are respectively denoted $S_A = [S_{A^-}, S_{A^+}]$ and K_A. Obviously, since S_A is an interval, we have:

$$S_{A^-} = M(S_A) - R(S_A) \quad \text{and} \quad S_{A^+} = M(S_A) + R(S_A) \quad (2)$$

In order to specify the fuzzy interval shape, two additional functions are used to link the support with the kernel value according to the vertical dimension. These functions, called left and right profiles and denoted respectively A^- (the increasing part) and A^+ (the decreasing part) for the fuzzy interval A, are defined as:

$$\begin{aligned} A^-(\lambda) &= Inf\ \{x \mid \mu_A(x) \geq \lambda\ ;\ x \geq S_{A^-}\} \\ A^+(\lambda) &= Sup\ \{x \mid \mu_A(x) \geq \lambda\ ;\ x \leq S_{A^+}\} \end{aligned} \quad (3)$$

where $\lambda \in [0, 1]$ represents the vertical dimension. It can be easily stated that:

$$K_A = A^-(1) = A^+(1), \qquad S_A = [A^-(0), A^+(0)]. \quad (4)$$

Finally, the fuzzy interval A can be univoquely defined by its left and right profiles. Thus, in the same way that the conventional interval a is denoted $[a^-, a^+]$, the fuzzy interval A will be denoted $[A^-, A^+]$.

Reciprocally, the profile representation $[A^-(\lambda), A^+(\lambda)]$ defines a unimodal fuzzy interval A if and only if the following conditions are satisfied:

(***a***) $A^-(\lambda)$ is increasing with respect to $\lambda \in [0, 1]$,
(***b***) $A^+(\lambda)$ is decreasing with respect to $\lambda \in [0, 1]$,
(***c***) $A^-(1) = A^+(1) = K_A$,
(***d***) $A^-(0) \leq K_A \leq A^+(0)$. (5)

It can be noted that condition (***d***) is induced by properties (***a***),(***b***) and (***c***).

3 Conventional Fuzzy Arithmetic Operations with Fuzzy Profiles

The conventional interval arithmetic operations $\{\oplus, \ominus, \otimes, \oslash\}$ defined for intervals [13][14] can be directly extended to fuzzy intervals using the left and right profiles instead of the lower and upper limits of the intervals [6][7].

3.1 Profile-based definitions

Let $A = [A^-(\lambda), A^+(\lambda)]$ and $B = [B^-(\lambda), B^+(\lambda)]$ be two fuzzy intervals, the classical four arithmetic operations are given by:

$$(A \oplus B)(\lambda) = [A^-(\lambda) + B^-(\lambda), A^+(\lambda) + B^+(\lambda)] \quad (6)$$

$$(A \ominus B)(\lambda) = [A^-(\lambda) - B^+(\lambda), A^+(\lambda) - B^-(\lambda)] \quad (7)$$

$$(A \otimes B)(\lambda) = [min\ Z(\lambda), max\ Z(\lambda)] \quad (8)$$

where $Z(\lambda) = \{A^-(\lambda).B^-(\lambda), A^-(\lambda).B^+(\lambda), A^+(\lambda).B^-(\lambda), A^+(\lambda).B^+(\lambda)\}$

For B such that $0 \notin S_B$

$$(A \oslash B)(\lambda) = [A^-(\lambda), A^+(\lambda)] \otimes [1/B^+(\lambda), 1/B^-(\lambda)] \quad (9)$$

3.2 Illustrative Example

For the sake of simplicity, let us consider two triangular fuzzy intervals A and B (see figure 3) given by:

$$A(\lambda) = [A^-(\lambda), A^+(\lambda)] = [1 + 2\lambda, 7 - 4\lambda]$$

$$B(\lambda) = [B^-(\lambda), B^+(\lambda)] = [-3 + \lambda, -1 - \lambda].$$

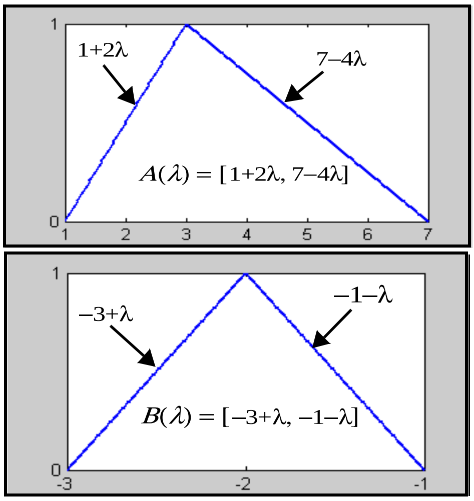


Figure 3: Fuzzy Intervals A and B.

Using the standard operations on the fuzzy profiles of A and B, the following results are obtained:

$$(A \oplus B)(\lambda) = [-2 + 3\lambda, 6 - 5\lambda] \quad (10)$$

$$(A \ominus B)(\lambda) = [2 + 3\lambda, 10 - 5\lambda] \quad (11)$$

$$(A \otimes B)(\lambda) = [-4\lambda^2 + 19\lambda - 21, -2\lambda^2 - 3\lambda - 1] \quad (12)$$

$$(A \oslash B)(\lambda) = [(7-4\lambda) / (-1-\lambda), (1+2\lambda) / (-3 + \lambda)] \quad (13)$$

The obtained results are illustrated in the following figures (see figure 4 and figure 5).

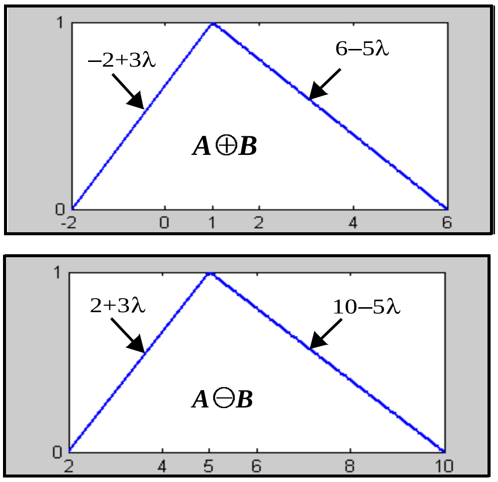


Figure 4: Addition and subtraction of A and B.

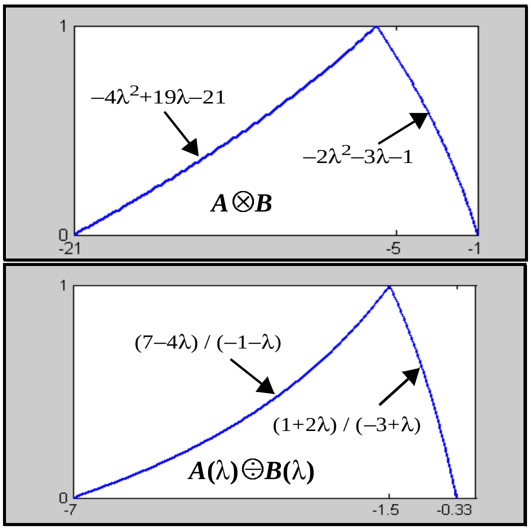


Figure 5: Multiplication and division of A and B.

3.3 Remarks

While being based on standard fuzzy arithmetics, these results are not controversial. However, as illustrated by the following computations:

$$(A\ominus A)(\lambda) = [-6 + 6\lambda,\ 6 - 6\lambda]$$
$$(A\oslash A)(\lambda) = [(1+2\lambda)/(7-4\lambda),\ (7-4\lambda)/(1+2\lambda)],$$

it is obvious that the defined operators produce counterintuitive results. Indeed, as mentioned in [19][21], $A\ominus A \neq 0$ and $A\oslash A \neq 1$, where 0 and 1 are singletons whose fuzzy representation is based on constant profiles, that is $\forall\lambda\in[0, 1]$, $0^-(\lambda) = 0^+(\lambda) = 0$ and $1^-(\lambda) = 1^+(\lambda) = 1$. It follows that the X solution of the fuzzy linear equation $A\oplus X = D$ is not, as we would expect, $X = D\ominus A$. For example, by choosing $D = [-2 + 3\lambda,\ 6 - 5\lambda]$, we would expect according to equation (10) that $X = D\ominus A$ be equal to B, i.e. $[-3+\lambda, -1-\lambda]$, which is unfortunately not the case:

$$X = D\ominus A = [-2+3\lambda,\ 6-5\lambda] \ominus [1+2\lambda,\ 7-4\lambda]$$
$$= [-9+7\lambda,\ 5-7\lambda] \neq B.$$

The same annoyance appears when solving the fuzzy equation $A\otimes X = E$ whose solution is not given by $X = E\oslash A$ as expected. For example, by choosing $E = [-4\lambda^2 + 19\lambda - 21,\ -2\lambda^2 - 3\lambda - 1]$ according to equation (12), the computation of $X = E\oslash A$ leads to a result different from B:

$$X = E\oslash A$$
$$= [-4\lambda^2 + 19\lambda - 21, -2\lambda^2 - 3\lambda - 1]\oslash[1+2\lambda, 7-4\lambda]$$
$$= [-4\lambda^2+19\lambda-21, -2\lambda^2-3\lambda-1]\otimes[1/7-4\lambda, 1/1+2\lambda]$$
$$= [-4\lambda^2 + 19\lambda - 21 / 1+2\lambda, -2\lambda^2 - 3\lambda - 1/ 7-4\lambda]$$
$$\neq B.$$

To sum up, the addition and subtraction (resp. multiplication and division) of fuzzy intervals are not reciprocal operations. According to this statement, it is not possible to solve inverse problems exactly using standard fuzzy arithmetic operators. In this context, the sequel of the paper is devoted to the definition of modified fuzzy arithmetic operators that allows exact inversion.

4 Exact Fuzzy Inverse Operators

Our objective is now to develop new subtraction and division operators denoted respectively $\ominus_\approx$ and $\oslash_\approx$ which are inverse operations of the addition $\oplus$ and multiplication $\otimes$ operators.

4.1 The modified $\ominus_\approx$ Operator

Definition 1: Let A and B be two unimodal fuzzy intervals with profile representation $A(\lambda) = [A^-(\lambda), A^+(\lambda)]$ and $B(\lambda) = [B^-(\lambda), B^+(\lambda)]$.

The modified difference operator $\ominus_\approx$ is defined by:

$$(A\ominus_\approx B)(\lambda) = [A^-(\lambda) - B^-(\lambda),\ A^+(\lambda) - B^+(\lambda)] \quad (14)$$
$$= [\Phi^-(\lambda), \Phi^+(\lambda)] = \Phi(\lambda)$$

when the result Φ actually represents a unimodal fuzzy interval.

Otherwise, $\ominus_\approx$ is undefined.

4.2 Properties of the operator $\ominus_\approx$

i. *Inverse operator of* $\oplus$

When defined, $\ominus_\approx$ is the inverse operator of $\oplus$, that is $B\oplus(A\ominus_\approx B) = (A\ominus_\approx B)\oplus B = A$.

Proof

$[B^-(\lambda), B^+(\lambda)]\oplus[\Phi^-(\lambda), \Phi^+(\lambda)]$

$= [B^-(\lambda), B^+(\lambda)]\oplus[A^-(\lambda) - B^-(\lambda), A^+(\lambda) - B^+(\lambda)]$

$= [A^-(\lambda), A^+(\lambda)] = A(\lambda)$.

ii. *Commutativity*

The operator $\ominus_\approx$ is not commutative, that is $A\ominus_\approx B \neq B\ominus_\approx A$.

iii. *Multiplication by a scalar*

$\omega \,.\, (A\ominus_\approx B) = (\omega \,.\, A \ominus_\approx \omega \,.\, B)$ for any scalar ω.

iv. *Neutral element*

The singleton 0 defined by a constant profile equal to 0 is a right neutral element of $\ominus_\approx$, that is: $A\ominus_\approx 0 = A$.

v. *Associativity*

The operator $\ominus_\approx$ is associative, that is:

$A\ominus_\approx B\ominus_\approx C = A\ominus_\approx(B\ominus_\approx C) = (A\ominus_\approx B)\ominus_\approx C$.

vi. *Inverse element*

Any fuzzy interval is its own inverse under the modified subtraction $\ominus_\approx$, i.e. $A\ominus_\approx A = 0$.

vii. *Regularity*

$A \ominus_{\approx} B = A \ominus_{\approx} C \quad \Rightarrow \quad B = C$.

viii. *Pseudo-distributivity with regard to* $\oplus$

$(A \oplus B) \ominus_{\approx} (C \oplus D) = (A \ominus_{\approx} C) \oplus (B \ominus_{\approx} D)$,

$(A \ominus_{\approx} B) \oplus (C \ominus_{\approx} D) = (A \oplus C) \ominus_{\approx} (B \oplus D)$.

4.3 Necessary existence condition for $\ominus_{\approx}$

According to equation (14), it is clear that $A \ominus_{\approx} B$ can always be computed. However, the difference operation is only valid when the obtained result $\Phi(\lambda)$ is a unimodal fuzzy interval, which means that the operator $\ominus_{\approx}$ exists only if $\Phi(\lambda)$ satisfies the four conditions expressed in (5). The aim of this subsection is to formulate a simple necessary existence condition of $\ominus_{\approx}$ that only depends on some features of the operands A and B.

Proposition 1: The operator $\ominus_{\approx}$ exists only if the following condition is satisfied:

$$| \Delta K - (M(S_A) - M(S_B)) | \leq R(S_A) - R(S_B) \qquad (15)$$

with : $\Delta K = K_A - K_B$.

Proof

$\Phi(\lambda)$ is an interval fuzzy number only if conditions (5) are verified.

According to (14), $\Phi^-(1) = A^-(1) - B^-(1)$ and $\Phi^+(1) = A^+(1) - B^+(1)$. As A and B are unimodal fuzzy intervals, it follows that $\Phi^-(1) = \Phi^+(1) = K_A - K_B = \Delta K$. So, condition (5)(***c***) always holds.

Satisfying condition (5)(***d***) requires that

$$\Phi^-(0) \leq \Delta K \leq \Phi^+(0) \quad \Leftrightarrow$$

$$A^-(0) - B^-(0) \leq \Delta K \leq A^+(0) - B^+(0) \quad \Leftrightarrow$$

$$(M(S_A) - R(S_A)) - (M(S_B) - R(S_B)) \leq \Delta K \leq (M(S_A) + R(S_A)) - (M(S_B) + R(S_B)) \quad \Leftrightarrow$$

$$- R(S_A) + R(S_B) \leq \Delta K - (M(S_A) - M(S_B)) \leq R(S_A) - R(S_B)) \quad \Leftrightarrow$$

$$| \Delta K - (M(S_A) - M(S_B)) | \leq R(S_A) - R(S_B),$$

which concludes the proof that condition (15) is a necessary condition for the definition of operator $\ominus_{\approx}$.

❑

Remarks

- It can be stated that the proposed difference operator is an adapted version of the Hukuhara difference definition for fuzzy intervals. Indeed, the Hukuhara difference of two sets $X \in C$ and $Y \in C$, if it exists, is a set $Z \in C$ such that $X = Y + Z$, where C is the family of all nonempty convex compact subsets [3][11].
- When triangular fuzzy numbers are considered, it can be easily proved that condition (15) is a sufficient condition for the existence of the operator $\ominus_{\approx}$. In other words, the satisfaction of (15) guarantees that Φ^- (resp. Φ^+) is increasing (resp. decreasing) with respect to $\lambda \in [0, 1]$.
- If the two fuzzy intervals A and B are symmetric, i.e. $K_A = M(S_A)$ and $K_B = M(S_B)$, then the existence condition (15) is reduced to the support condition $R(S_A) - R(S_B) \geq 0$.

4.4 The modified $\oslash_{\approx}$ Operator

Definition 2: Let A and B be two unimodal fuzzy intervals with $0 \notin S_B$, the modified division operator $\oslash_{\approx}$ is defined by:

$$A(\lambda) \oslash_{\approx} B(\lambda) = \Phi(\lambda) = [\Phi^-(\lambda), \Phi^+(\lambda)] \qquad (16)$$

where:

$$\Phi^-(\lambda) = \frac{Num^-(\lambda)}{Den^-(\lambda)} = \frac{M(A(\lambda)) - R(A(\lambda)) \cdot sign(M(S_B))}{M(B(\lambda)) - R(B(\lambda)) \cdot sign(Num^-(\lambda))} \qquad (17)$$

and

$$\Phi^+(\lambda) = \frac{Num^+(\lambda)}{Den^+(\lambda)} = \frac{M(A(\lambda)) + R(A(\lambda)) \cdot sign(M(S_B))}{M(B(\lambda)) + R(B(\lambda)) \cdot sign(Num^+(\lambda))} \qquad (18)$$

when the result Φ actually represents a unimodal fuzzy interval.

Otherwise, $\oslash_{\approx}$ is undefined.

4.5 Properties of the operator $\oslash_{\approx}$

i. *Inverse operator of* $\otimes$

When defined, $\oslash_{\approx}$ is the inverse operator of $\otimes$, that is $B \otimes (A \oslash_{\approx} B) = (A \oslash_{\approx} B) \otimes B = A$.

Proof

$$B(\lambda) \otimes (A(\lambda) \oslash_{\approx} B(\lambda)) = B(\lambda) \otimes \Phi(\lambda) = [(B(\lambda) \otimes \Phi(\lambda))^-, (B(\lambda) \otimes \Phi(\lambda))^+] \qquad (19)$$

By substituting the expression of Φ given by (17) and (18) in equation (19) and according to the sign of S_B, two cases can be distinguished:

1) If $S_B > 0$, the expression is given by:

$(B \otimes \Phi)^- = M(B).\Phi^- - R(B).\Phi^-. sign(\Phi^-)$

$(B \otimes \Phi)^+ = M(B).\Phi^+ + R(B).\Phi^+. sign(\Phi^+)$

otherwise expressed as:

$$(B \otimes \Phi)^{-} = \frac{M(B) \cdot A^{-}}{M(B) - R(B) \cdot sign(Num^{-})} - \frac{R(B) \cdot A^{-}}{M(B) - R(B) \cdot sign(Num^{-})} \cdot sign(\Phi^{-}) \quad (20)$$

and

$$(B \otimes \Phi)^{+} = \frac{M(B) \cdot A^{+}}{M(B) + R(B) \cdot sign(Num^{+})} + \frac{R(B) \cdot A^{+}}{M(B) + R(B) \cdot sign(Num^{+})} \cdot sign(\Phi^{+}) . \quad (21)$$

As $sign(Num^{-}) = sign(A^{-}) = sign(\Phi^{-})$
$sign(Num^{+}) = \mathrm{sign}(A^{+}) = sign(\Phi^{+})$
equations (20) and (21) can be rewritten as:

$$(B \otimes \Phi)^{-} = \frac{(M(B) - R(B) \cdot sign(\Phi^{-})) \cdot A^{-}}{(M[B] - R[B] \cdot sign(\Phi^{-}))} = A^{-}$$
$$(B \otimes \Phi)^{+} = \frac{(M(B) + R(B) \cdot sign(\Phi^{+})) \cdot A^{+}}{(M(B) + R(B) \cdot sign(\Phi^{+}))} = A^{+} \quad (22)$$

2) If $S_B < 0$, it can be stated that:
$(B \otimes \Phi)^{-} = M(B).\Phi^{+} - R(B).\Phi^{+}.\ sign(\Phi^{+})$,
$(B \otimes \Phi)^{+} = M(B).\Phi^{-} + R(B).\Phi^{-}.\ sign(\Phi^{-})$.

By adopting the same principle as in the previous case, the following expression are obtained:

$$(B \otimes \Phi)^{-} = A^{-}. \qquad (B \otimes \Phi)^{+} = A^{+}. \quad (23)$$

❑

ii. *Commutativity*

The operator $\ominus_{\approx}$ is not commutative, that is $A \ominus_{\approx} B \neq B \ominus_{\approx} A$.

iii. *Multiplication by a scalar*

$\omega.(A \ominus_{\approx} B) = (\omega . A \ominus_{\approx} \omega . B)$ for any scalar ω.

iv. *Neutral element*

The singleton 1 defined by a constant profile equal to 1 is a right neutral element of $\ominus_{\approx}$, that is $A \ominus_{\approx} 1 = A$.

v. *Associativity*

The operator $\ominus_{\approx}$ is associative, i.e.
$A \ominus_{\approx} B \ominus_{\approx} C = A \ominus_{\approx} (B \ominus_{\approx} C) = (A \ominus_{\approx} B) \ominus_{\approx} C$.

vi. *Inverse element*

Any fuzzy interval is its own inverse under the modified division $\ominus_{\approx}$, i.e. $A \ominus_{\approx} A = 1$.

vii. *Regularity*

$A \ominus_{\approx} B = A \ominus_{\approx} C \Rightarrow B = C$.

viii. *Distributivity with regard to* ⊕

$(A \oplus B) \ominus_{\approx} C = (A \ominus_{\approx} C) \oplus (B \ominus_{\approx} C)$.

4.6 Necessary existence condition for $\ominus_{\approx}$

Proposition 2: The operator $\ominus_{\approx}$ exists only if the following condition is satisfied:

$$\frac{1 - |Rex(S_A)|}{1 - S \cdot |Rex(S_B)|} \leq \frac{\delta K}{\delta M} \leq \frac{1 + |Rex(S_A)|}{1 + |Rex(S_B)|} \quad (24)$$

where $\delta K = K_A / K_B$, $\delta M = M(S_A) / M(S_B)$,
$Rex(.) = R(.) / M(.)$ and $S = sign(1 - |Rex(S_A)|)$.

Proof

$\Phi(\lambda)$ is an interval fuzzy number only if conditions (5) are verified.

According to (17), as $R(A(1)) = R(B(1)) = 0$,
$\Phi^{-}(1) = M(A(1)) / M(B(1)) = K_A / K_B = \delta K$

In the same way, according to (18),
$\Phi^{+}(1) = M(A(1)) / M(B(1)) = K_A / K_B = \delta K$

So, $\Phi^{-}(1) = \Phi^{+}(1) = \delta K$, which means that continuity between left and right profiles is guaranteed. In other words, condition (5)(***c***) always holds.

Satisfying condition (5)(***d***) requires that

$$\Phi^{-}(0) \leq \delta K \leq \Phi^{+}(0) \quad (25)$$

Two situations have to be considered according to the sign of B that is:

1) If B is positive, according to equations (17) and (18), the inequality (25) becomes:

$$\delta M \frac{1 - Rex(S_A)}{1 - s_1 Rex(S_B)} \leq \delta K \leq \delta M \frac{1 + Rex(S_A)}{1 + s_2 Rex(S_B)} \quad (26)$$

where $s_1 = sign\ (M(S_A) - R(S_A))$ and
$s_2 = sign\ (M(S_A) + R(S_A))$.

2) If B is negative, the inequality (25) becomes:

$$\delta M \frac{1 + Rex(S_A)}{1 - s_2 Rex(S_B)} \leq \delta K \leq \delta M \frac{1 - Rex(S_A)}{1 + s_1 Rex(S_B)} . \quad (27)$$

By regrouping inequalities (26) and (27), it follows:

$$\frac{1 - |Rex(S_A)|}{1 - s_1 s_2 |Rex(S_B)|} \leq \frac{\delta K}{\delta M} \leq \frac{1 + |Rex(S_A)|}{1 + |Rex(S_B)|} . \quad (28)$$

As $s_1 s_2 = sign\ (M(S_A)^2 - R(S_A)^2) = sign\ (1 - Rex(S_A)^2) = sign\ (1 - |Rex(S_A)|) = S$, it follows that condition (5)(***d***) is satisfied if and only if constraint (24) holds.

❑

Remark

- The inequality (24) is a necessary condition for the existence of $A \ominus_{\approx} B$ but not a sufficient one. Whenever condition (24) holds, the existence of $A \ominus_{\approx} B$ further required that Φ^{-} (resp. Φ^{+}) be increasing (resp. decreasing) with respect to $\lambda \in [0, 1]$. Actually, it is possible that condition (24) is verified for all the level-cuts of A and B without Φ being a fuzzy subset. In other words, the monotonicity of Φ^- and Φ^+ not only depends on the support and kernel of A and B but also on their shape. For example, it can be verified that condition (24) is sufficient for triangular fuzzy intervals A and B. On the contrary, when A is the product of two triangular fuzzy intervals, condition (24) is no more sufficient.

4.7 Illustrative Example

Case 1: Let us illustrate the behavior of the proposed modified operators for computing with the two fuzzy intervals A and B previously defined in section 3.2

It can be verified that the condition (15) is respected, i.e.

$$\Delta K = K_A - K_B = 3 - (-2) = 5$$

$$M(S_A) - M(S_B) = M([1,7]) - M([-3,-1]) = 6$$

$$R(S_A) - R(S_B) = 3 - 1 = 2$$

$$|\,\Delta K - (M(S_A) - M(S_B))\,| = |5 - 6| \leq R(S_A) - R(S_B) = 2$$

In this case, the modified difference between A and B is defined and given by:

$$A(\lambda) \ominus_{\approx} B(\lambda) = [1+2\lambda, 7-4\lambda] \ominus_{\approx} [-3+\lambda, -1-\lambda] = [4+\lambda, 8-3\lambda]$$

As expected, it can be verified that:

$$[\,4+\lambda, 8-3\lambda\,] \oplus [\,-3+\lambda, -1-\lambda\,] = A(\lambda)$$

In the same way, let us evaluate $A \oslash_{\approx} B$. Firstly let us verify that the existence condition (24) is respected.

$$\delta K = K_A / K_B = 3/(-2) = -6$$

$$\delta M = M(S_A) / M(S_B) = 4/(-2) = -8$$

By substituting these values into (24), it follows:

$$0.5 \leq \frac{6}{8} \leq \frac{7}{6},$$

which means that (24) is verified.

Then, computing $A \oslash_{\approx} B$, we get:

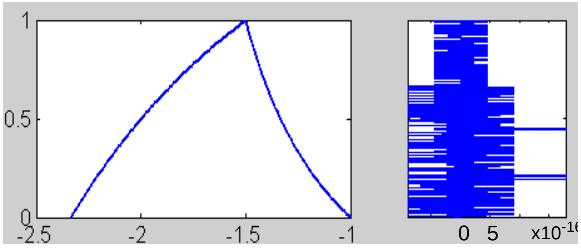
$$[1+2\lambda, 7-4\lambda] \oslash_{\approx} [-3+\lambda, -1-\lambda] = [\Phi^-(\lambda), \Phi^+(\lambda)]$$

with:

$$\Phi^-(\lambda) = \frac{(4-\lambda) - (3-3\lambda)\cdot(-1)}{(-2) - (1-\lambda)\cdot(1)} = \frac{7-4\lambda}{-3+\lambda}$$

$$\Phi^+(\lambda) = \frac{(4-\lambda) + (3-3\lambda)\cdot(-1)}{(-2) + (1-\lambda)\cdot(1)} = \frac{1+2\lambda}{-1-\lambda}$$

It can be verified that $\oslash_{\approx}$ is the exact inverse of $\otimes$ as shown in Figure 6 where the right plot illustrates that $A \ominus_{\approx} [(A \oslash_{\approx} B) \otimes B] = 0$ (with some computation noise).



$A \oslash_{\approx} B$ $\quad$ $A \ominus_{\approx} [(A \oslash_{\approx} B) \otimes B]$

Figure 6: Exact fuzzy inverse operators

Case 2: Let us now consider two other fuzzy intervals A and B:

$$A(\lambda) = [A^-(\lambda), A^+(\lambda)] = [1 + 0.5\lambda, 5 - 3.5\lambda]$$

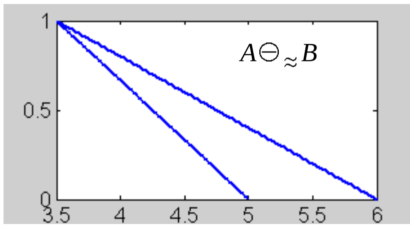
$$B(\lambda) = [B^-(\lambda), B^+(\lambda)] = [-4 + 2\lambda, -1 - \lambda].$$

It can be stated that condition (15) is violated. Indeed, $|3.5 - 5.5| \geq 0.5$. In this case, the difference operator is undefined. Actually, computing $A \ominus_{\approx} B$ would produce $[5 - 1.5\lambda, 6 - 2.5\lambda]$, that is a non valid result since it does not represent a fuzzy interval (see Figure 7(a)).

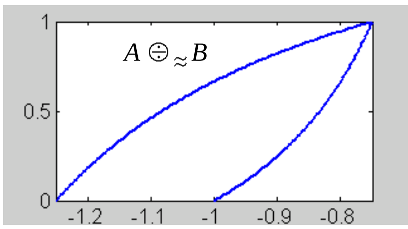
In the same way, it can be verified that the existence condition for the division operator is not respected:

$$\frac{5}{6} \leq \frac{10}{9} \geq \frac{50}{48}.$$

By computing the result that would be obtained, we get $[(5 - 3.5\lambda) / (-4 + 2\lambda), (1 + 0.5\lambda) / (-1 - \lambda)]$, again a non valid result as illustrated by Figure 7(b).



(a) Subtraction



(b) Division

Figure 7: Non valid operations

5 Conclusion

This paper presents a new methodology for the implementation of the substraction and the division operators between fuzzy intervals with their existence conditions. The proposed operators are exact inverses of the addition and multiplication operators. In some cases, these new modified operators can solve the overestimation problem well known in fuzzy arithmetics.

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