An algebraic approach to states on MV-algebras

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Abstract

In this paper we will introduce the variety of *state MV-algebras* by enlarging the language of MV-algebras with an unary operator σ, and by adding equations ensuring σ to satisfy some basic properties of a *state* on an MV-algebra in the sense of [4]. The main result of this paper says us that there is a standard way to define a state MV-algebra starting from a state on an MV-algebra and vice-versa, each state MV-algebra defines a state on an MV-algebra in its usual meaning.

Keywords: MV-algebra, state, tensor product, coherence.

1 Introduction

MV-algebras was introduced by Chang in [2] in order to show Łukasiewicz logic (see [3]) to be standard complete, i.e. complete with respect to evaluations of propositional variables in the real unit interval $[0,1]$. Chang's Completeness Theorem says us that the real unit interval $[0,1]$ is the standard domain of interpretation for the propositional variables of Łukasiewicz logic. Recall in fact that the prototypical example of an MV-algebra is the standard one

$$[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle,$$

where for all $x, y \in [0,1]$, $x \oplus y = \min\{1, x+y\}$, and $\neg x = 1 - x$.

MV-algebras provide a generalization of Boolean algebras in the sense that each Booelan algebra is an MV-algebra satisfying the equation $x \oplus x = x$. Therefore as to generalize probability on Boolean algebras, Mundici introduced in [4] the notion of state on MV-algebras. Roughly speaking a state on an MV-algebra $\mathcal{M}$ is a (finitely) additive (in the sense of $\mathcal{M}$) function $s : \mathcal{M} \to [0,1]$ mapping the bottom element of $\mathcal{M}$ in 0. Hence each state s maps element of a given MV-algebra $\mathcal{M}$ into the domain of the standard MV-algebra $[0,1]_{MV}$, for this reason it makes sense to consider a state as an additional operator on $\mathcal{M}$.

Following this idea, in this paper we will enlarge the language of MV-algebras by means of an unary operation σ and we will add, to those ones of MV-algebras, a class of equations ensuring σ to satisfy some basic properties of states. The resulting algebraic structure will be called *state MV-algebra*. These algebras will be introduced in Section 3, while in Section 5 we will show how each state MV-algebra induce (at least) a state (in its the usual meaning) on its MV-reduct, while each state on an MV-algebra $\mathcal{M}$ defines a state MV-algebra whose MV-reduct is the tensor product (see [5]) of $[0,1]_{MV}$ by $\mathcal{M}$. In Section 6 we will apply state MV-algebras to study the probabilistic coherence problem on infinite-valued events. We will end with some remarks and open problems.

2 Preliminaries

Definition 2.1 *An MV-algebra is an algebra* $\mathcal{M} = \langle M, \oplus, \neg, 0, 1 \rangle$ *such that:*

- $\langle M, \oplus, 0 \rangle$ *is a commutative monoid, having* 0 *as neutral element,*
- $\neg\neg x = x$ *and* $x \oplus 1 = 1$ *hold for all* $x \in M$*,*
- $x \oplus \neg(x \oplus \neg y) = y \oplus \neg(y \oplus \neg x)$ *is satisfied for all* $x, y \in M$*.*

In any MV-algebra $\mathcal{M}$ one can define the following further operations:

$$x \to y = \neg x \oplus y,\ x \ominus y = \neg(x \to y),\ x \odot y = \neg(\neg x \oplus \neg y),\ x \leftrightarrow y = (x \to y) \odot (y \to x),\ x \vee y = (x \to y) \to y,\ x \wedge y = \neg(\neg x \vee \neg y).$$

Finally any MV-algebra $\mathcal{M}$ is equipped with the order relation: $x \leq y$ iff $x \to y = 1$. An MV-algebra $\mathcal{M}$

is said *linearly ordered* iff the order relation defined as above is linear. Following the tradition, a linearly ordered MV-algebra will be also called an *MV-chain.*

Definition 2.2 ([4]) *A* state *on an MV-algebra $\mathcal{M}$ is a mapping $s : M \to [0,1]$ such that:*

- $s(1) = 1$,
- *if $x \odot y = 0$ for $x, y \in M$, then $s(x \oplus y) = s(x) + s(y)$.*

It is quite easy to see that each state s on an MV-algebra $\mathcal{M}$ is monotone and satisfies $s(0) = 0$. A state $s : M \to [0,1]$ is said *faithfull* if $s(x) > 0$, whenever $x \neq 0$.

3 State MV-algebras

Definition 3.1 *A* state MV-algebra *is a pair $(\mathcal{M}, \sigma)$ such that $\mathcal{M}$ is an MV-algebra and σ is a unary operation on M satisfying:*
(1) $\sigma(1) = 1$,
(2) $\sigma(\neg x) = \neg\sigma(x)$,
(3) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$.
(4) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.

Obviously the class of state MV-algebras forms a variety. In the following Lemma we will prove some properties of state MV-algebras.

Lemma 3.2 *In a state MV-algebra $(\mathcal{M}, \sigma)$ the following conditions hold:*
(a) $\sigma(0) = 0$.
(b) *If $x \leq y$, then $\sigma(x) \leq \sigma(y)$.*
(c) $\sigma(x \oplus y) \leq \sigma(x) \oplus \sigma(y)$, *and if $x \odot y = 0$, then $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$.*
(d) $\sigma(x \ominus y) \geq \sigma(x) \ominus \sigma(y)$ *and if $y \leq x$, then $\sigma(x \ominus y) = \sigma(x) \ominus \sigma(y)$.*
(e) *Letting $d(x,y) = (x \ominus y) \oplus (y \ominus x)$, we have that $d(\sigma(x), \sigma(y)) \leq \sigma(d(x,y))$.*
(f) $\sigma(x) \odot \sigma(y) \leq \sigma(x \odot y)$. *Thus if $x \odot y = 0$, then $\sigma(x) \odot \sigma(y) = 0$.*
(g) $\sigma(\sigma(x)) = \sigma(x)$.
(h) *The image $\sigma(M)$ of M under σ is the domain of a MV-subalgebra of $\mathcal{M}$.*

Proof. (a) By (1) and (2).

(b) If $x \leq y$, then $y = x \oplus (y \ominus x)$, therefore since $x \odot (y \ominus x) = 0$, substituting in (2) $(y \ominus x)$ for y, we get $\sigma(y) = \sigma(x \oplus (y \ominus x)) = \sigma(x) \oplus \sigma(y \ominus x) \geq \sigma(x)$.

(c) By (b), $\sigma(y) \geq \sigma(y \ominus (x \odot y))$, therefore $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y)) \leq \sigma(x) \oplus \sigma(y)$. If $(x \odot y) = 0$, then $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y)) = \sigma(x) \oplus \sigma(y)$.

(d) Using (2) and (c), we obtain:

$\sigma(x \ominus y) = \sigma(\neg(\neg x \oplus y)) = \neg\sigma(\neg x \oplus y) \geq \neg(\sigma(\neg x) \oplus \sigma(y)) =$

$= \neg(\neg\sigma(x) \oplus \sigma(y)) = \sigma(x) \ominus \sigma(y)$. Moreover, if $y \leq x$, then $\neg x \odot y = 0$, and by (c), $\sigma(x \ominus y) = \neg\sigma(\neg x \oplus y) = \neg(\sigma(\neg x) \oplus \sigma(y)) = \sigma(x) \ominus \sigma(y)$.

(e) Since $(x \ominus y) \odot (y \ominus x) = 0$, by (d) and (c) we get

$\sigma(d(x,y)) = \sigma(x \ominus y) \oplus \sigma(y \ominus x) \geq (\sigma(x) \ominus \sigma(y)) \oplus (\sigma(y) \ominus \sigma(x)) = d(\sigma(x), \sigma(y))$.

(f) We have $x \odot y = x \ominus \neg y$, therefore by (e), $\sigma(x \odot y) \geq \sigma(x) \ominus \sigma(\neg y) = \sigma(x) \ominus (\neg\sigma(y)) = \sigma(x) \odot \sigma(y)$. If $x \odot y = 0$, then $0 = \sigma(x \odot y) \geq \sigma(x) \odot \sigma(y)$, therefore $\sigma(x \odot y) = 0$.

(g) By (a), $\sigma(0) = 0$, therefore using (4) we get $\sigma(\sigma(x)) = \sigma(\sigma(x) \oplus \sigma(0)) = \sigma(x) \oplus \sigma(0) = \sigma(x)$.

(h) By (g), the range of σ is constituted by all the fixed points of σ, therefore it is sufficient to prove that the set of such fixed points is closed under $\oplus$ and under $\neg$. Now closure under $\oplus$ follows from (4). As regards to closure under $\neg$, using (2) and (g), we get $\sigma(\neg\sigma(x)) = \neg(\sigma(\sigma(x)) = \neg(\sigma(x))$, and the claim follows. ■

Examples. (a) We start from a trivial example. Let $\mathcal{M}$ be any MV-algebra and σ be the identity on M. Then $(\mathcal{M}, \sigma)$ is a state MV-algebra.

(b) A slightly less silly example. Let σ be an idempotent endomorphism of an MV-algebra $\mathcal{M}$ (For example, we may take $\mathcal{M}$ to be a non-trivial ultrapower of the standard MV-algebra $[0,1]_{MV}$ and σ to be the standard part function). Then $(\mathcal{M}, \sigma)$ is a state MV-algebra.

(c) This is in our opinion a sufficiently general example. Let $\mathcal{M}$ be the algebra of all continuous and piecewise linear functions with real coefficients from $[0,1]^n$ into $[0,1]$. Then $\mathcal{M}$, with the point-wise application of Łukasiewicz $\oplus$ and $\neg$, forms an MV-algebra. Now let for $f \in M$, $\sigma(f) = \int_{[0,1]^n} f$. Then $(\mathcal{M}, \sigma)$ is a state MV-algebra. Note that $(\mathcal{M}, \sigma)$ is simple, therefore it is subdirectly irreducible, but it is not totally ordered. Although rather general, this algebra satisfies the quasi equation $\sigma(x) = 0$ implies $x = 0$, which is not valid in general.

4 Congruences, ideals, subdirectly irreducible elements

A σ-ideal of a state MV-algebra $(\mathcal{M}, \sigma)$ is an MV-ideal closed under σ. Since $\sigma(0) = 0$, the congruence classes of 0 are σ-ideals. Conversely, given a

σ ideal J of a state MV-algebra $(\mathcal{M}, \sigma)$, the relation $\theta_J = \{(x, y) : d(x, y) \in J\}$ is a congruence of $(\mathcal{M}, \sigma)$. That θ_J is a congruence of $\mathcal{M}$ is well-known. Moreover, if $(x, y) \in \theta_J$, then $d(x, y) \in J$, therefore $\sigma(d(x, y)) \in J$, as J is a σ-ideal. Moreover by Lemma 3.2, (e) $d(\sigma(x), \sigma(y)) \leq \sigma(d(x, y))$. Thus $d(\sigma(x), \sigma(y)) \in J$, and $(\sigma(x), \sigma(y)) \in \theta_J$. Thus θ_J is a congruence. Therefore:

Theorem 4.1 *The congruence lattice of a state MV-algebra is isomorphic to the lattice of its σ-ideals.*

Lemma 4.2 *The σ-ideal J_a generated by an element a of a state MV-algebra $(\mathcal{M}, \sigma)$ is $\{x : \exists n \in \omega(x \leq n(a \oplus \sigma(a)))\}$.*

Proof. Let $I = \{x : \exists n \in \omega(x \leq n(a \oplus s(a)))\}$. By the definition of σ-ideal, we have that every element of I must be in J_a. Thus $I \subseteq J_a$. For the opposite inclusion, it suffices to show that I is an ideal which contains a. That $a \in I$ is clear, and the proof that I is an MV-ideal is routine. We show that I is closed under σ. By Lemma 3.2, (c) and (g), if $x \leq n(a \oplus \sigma(a))$, then $\sigma(x) \leq n(\sigma(a) \oplus \sigma(a)) \leq 2n(a \oplus \sigma(a))$, therefore $\sigma(x) \in J$. This ends the proof. ■

Theorem 4.3 *(a) If $(\mathcal{M}, \sigma)$ is a subdirectly irreducible, then $\sigma(M)$ is linearly ordered.*
(b) If $(\mathcal{M}, \sigma)$ satisfies the quasi identity $\sigma(x) = 0$ implies $x = 0$, then $(\mathcal{M}, \sigma)$ is subdirectly irreducible iff $\sigma(M)$ is subdirectly irreducible (as an MV-algebra).

Proof. (a) Let J be the minimum non-trivial σ-ideal of $\mathcal{M}$. By its minimality, J is mono-generated, therefore there is $a > 0$ such that $J = J_a = \{x : \exists n \in \omega(x \leq n(a \oplus \sigma(a)))\}$. Suppose that for some $\sigma(x), \sigma(y) \in \sigma(M)$ one has $\sigma(x) \not\leq \sigma(y)$ and $\sigma(y) \not\leq \sigma(x)$. Then the ideals $J_{\sigma(x)\ominus\sigma(y)}$ and $J_{\sigma(y)\ominus\sigma(x)}$ generated by $\sigma(x) \ominus \sigma(y)$ and by $\sigma(y) \ominus \sigma(x)$ respectively are non trivial, therefore they both contain J. Since $\sigma(M)$ is closed under σ, by Lemma 4.2, there is $n \in \omega$ such that $a \leq n((\sigma(x) \ominus \sigma(y)))$ and $a \leq n(\sigma(y) \ominus \sigma(x))$ (here we are using in a crucial way the fact that σ is the identity on $\sigma(M)$ and that $\sigma(M)$ is a subalgebra of A, cf Lemma 3.2 (g) and8 (h)). Hence $a \leq n((\sigma(x) \ominus \sigma(y)) \wedge (\sigma(y) \ominus \sigma(x))) = 0$, and a contradiction has been reached.

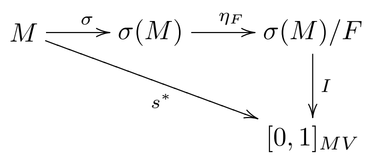
(b) If $\sigma(x) = 0$ implies $x = 0$, then the intersection of a non-trivial σ ideal J of $(\mathcal{M}, \sigma)$ with $\sigma(M)$ is a non-trivial ideal of $\sigma(M)$. Moreover any ideal of $\sigma(M)$ is closed under σ, therefore in $\sigma(M)$ every ideal is a σ ideal. Hence if J is the minimum non-trivial σ ideal of $\sigma(M)$, then $J \cap \sigma(M)$ is the minimum non-trivial ideal of $\sigma(M)$. Indeed if I is another non-trivial ideal of $\sigma(M)$, then the σ ideal H of $(\mathcal{M}, \sigma)$ generated by I contains J, and $I = H \cap \sigma(A) \supseteq J \cap \sigma(A)$. It follows that $J \cap \sigma(M)$ is minimal. Hence if $(\mathcal{M}, \sigma)$ is subdirectly irreducible, then so is $\sigma(M)$.

Conversely, if I is the minimum non-trivial ideal of $\sigma(M)$, then the σ ideal K of $(\mathcal{M}, \sigma)$ generated by I is the minimum non-trivial σ-ideal of $(\mathcal{M}, \sigma)$. Indeed, if J is another non-trivial σ ideal of $(\mathcal{M}, \sigma)$, then $J \cap \sigma(M) \supseteq K \cap \sigma(M) = I$. Thus K contains the σ ideal generated by I, i.e., $K \supseteq J$ and J is minimal. Hence if $\sigma(M)$ is subdirectly irreducible, then so is $(\mathcal{M}, \sigma)$. ■

Remark. As already noted, not all subdirectly irreducible state MV-algebras are linearly ordered. Moreover, linearly ordered state MV-algebras satisfy the identity $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$, which is not valid in general. Thus the variety of state MV-algebras is not generated by its totally ordered members. However, by Theorem 4.3, it is generated as a quasivariety by its member $(\mathcal{M}, \sigma)$ such that $\sigma(M)$ is totally ordered.

5 State MV-algebras and states on MV-algebras

Let $(\mathcal{M}, \sigma)$ be a state MV-algebra, and let F be a maximal MV-filter over $\sigma(M)$. Also, let for $x \in M$, $s^*(x) = \sigma(x)/F$. Note that $\sigma(M)/F$ (considered as an MV-algebra) is a simple MV-algebra, therefore it embeds into the standard MV-algebra $[0, 1]_{MV}$ on $[0, 1]$. Thus s^* can be regarded as a map from $\mathcal{M}$ into $[0, 1]_{MV}$.



η_F being the quotient map of $\sigma(M)$ in the simple MV-algebra $\sigma(M)/F$, and I being the embedding of $\sigma(M)/F$ into the standard MV-algebra $[0, 1]_{MV}$.

Theorem 5.1 *s^* is a state on $\mathcal{M}$.*

Proof. That $s^*(1) = 1$ is clear. Now let $x, y \in M$ such that $x \odot y = 0$. Then by Lemma 3.2 (c) and (f), we get that $s^*(x \oplus y) = s^*(x) \oplus s^*(y)$ and $\sigma(x) \odot \sigma(y) = 0$. Hence $s^*(x) \oplus s^*(y) = s^*(x) + s^*(y)$, and $s^*(x \oplus y) = s^*(x) + s^*(y)$. Thus s^* is a state. ■

Now let s be a state on a MV-algebra $\mathcal{M}$. Consider the tensor product $\mathcal{T} = [0, 1]_{MV} \otimes \mathcal{M}$, (see Mundici's paper [5]). Every element of $\mathcal{T}$ can be written as $\alpha \otimes a$ with $\alpha \in [0, 1]$ and $a \in M$. (Warning: different expressions may denote the same element, for instance, for any $\alpha \in [0, 1]$ and $a \in M$, both $\alpha \otimes 0$ and $0 \otimes a$ denote the bottom element of $\mathcal{T}$. We also warn the reader that $\otimes$ is not an operation on $\mathcal{T}$, but a function from $[0,1] \times M$ into $\mathcal{T}$). Moreover, for $a, b, c, d \in M$ and $\alpha, \beta, \gamma, \delta \in [0,1]$, the following conditions are satisfied:

(i) $\mathcal{T}$ is a MV-algebra.

(ii) $(\alpha \oplus \beta) \otimes 1 = (\alpha \otimes 1) \otimes (\beta \otimes 1)$ and $(1 \otimes (a \oplus b)) = (1 \otimes a) \oplus (1 \otimes b)$.

(iii) $\neg\alpha \otimes 1 = \neg(\alpha \otimes 1)$ and $1 \otimes \neg a = \neg(1 \otimes a)$.

(iv) The maps $\alpha \mapsto \alpha \otimes 1$ and $a \mapsto 1 \otimes a$ are embeddings of $[0,1]_{MV}$ and of $\mathcal{M}$ respectively into $\mathcal{T}$. Thus we will identify α with $\alpha \otimes 1$ and a with $1 \otimes a$.

(v) If $\beta \odot \gamma = 0$ and $b \odot c = 0$, then $(\beta + \gamma) \otimes a = (\beta \otimes a) \oplus_T (\gamma \otimes a)$ and $\alpha \otimes (b \oplus c) = (\alpha \otimes c) \oplus_T (\alpha \otimes d)$.

(vi) $\alpha \otimes (a \ominus d) = (\alpha \otimes a) \ominus_T (\alpha \otimes d)$ and $(\alpha \ominus \delta) \otimes a = (\alpha \otimes a) \ominus_T (\delta \otimes a)$.

The top element of $\mathcal{T}$ is $1 = 1 \otimes 1$, and the bottom element is $0 \otimes 0 = 0 \otimes a = \alpha \otimes 0$.

We define an operation σ^+ on $\mathcal{T}$ as follows: $\sigma^+(\alpha \otimes a) = \alpha \cdot s(a)$ (note that $\alpha \cdot s(a) \in [0,1]_{MV}$ which is a subalgebra of $\mathcal{T}$).

Theorem 5.2 *σ^+ is well-defined and $(\mathcal{T}, \sigma^+)$ is a state MV-algebra.*

Proof. Let $\mathcal{M}^d$ be the divisible MV-algebra generated by $\mathcal{M}$. Hence every element of $\mathcal{M}^d$ has the form qa for some rational $q \in [0,1]$ and $a \in M$. There is only one way to extend $s : \mathcal{M} \to [0,1]$ to a state s^d on $\mathcal{M}^d$ into $[0,1]$. In fact it must be that $s^d(qa) = qs(a)$. Let now $\mathcal{T}_{\mathbb{Q}}$ be the MV-algebra $[0,1]_{MV} \cap \mathbb{Q} \otimes \mathcal{M}$. The following holds:

Claim 5.3 *$\mathcal{T}_{\mathbb{Q}}$ is isomorphic to $\mathcal{M}^d$ via the isomorphism $f : q \otimes a \mapsto qa$.*

Proof. (Claim) Let $\beta : [0,1] \cap \mathbb{Q} \times M \to M^d$ be defined as $\beta(q,a) = qa$. Then β is a bimorphism (see [5]). In fact the following holds:

- $\beta(1,1) = 1 \cdot 1 = 1$,
- $\beta(0,a) = \beta(q,0) = 0$,
- $\beta(q, a_1 \vee a_2) = q(a_1 \vee a_2) = qa_1 \vee qa_2 = \beta(q,a_1) \vee \beta(q,a_2)$, and analogously the other cases.
- If $a_1 \odot a_2 = 0$, then
  - $\beta(q,a_1) \odot \beta(q,a_2) = qa_1 \odot qa_2 \le a_1 \odot a_2 = 0$.
  - $\beta(q, a_1 \oplus a_2) = q(a_1 \oplus a_2)$. But $\cdot$ distributes on $\oplus$, and thus $\beta(q, a_1 \oplus a_2) = \beta(q,a_1) \oplus \beta(q,a_2)$

  Symmetrically the other cases when $q_1 \odot q_2 = 0$ hold.

Therefore β is a bimorphism. Now from [5] (Theorem 3.2 pp. 234–235), there exists a unique homomorphism $\lambda : [0,1]_{MV} \cap \mathbb{Q} \otimes \mathcal{M} \to \mathcal{M}^d$. Moreover it is easy to see that λ is a bijection. In fact if $\lambda(q \otimes a) \neq \lambda(r \otimes b)$, then $qa \neq rb$, and thus $q \otimes a \neq r \otimes b$, then λ is injective. To prove that λ is surjective is trivial. In fact notice that for each $qa \in A^d$, at least $\lambda(q \otimes a) = qa$. Thus the claim holds. ■

Therefore $q \otimes a$ can be identified with $qa = 1 \otimes qa$, and $\sigma^+(q \otimes a) = qs(a) = s^d(qa)$. It follows that if for some rationals q, r and for some $a, b \in M$ one has $r \otimes a = q \otimes b$, then $ra = qb$ and $\sigma^+(r \otimes a) = rs(a) = s^d(ra) = s^d(qb) = qs(b) = \sigma^+(q \otimes b)$. Thus σ^+ is well-defined on $\mathcal{M}^d$.

Now the restriction of σ^Q on $\mathcal{T}_{\mathbb{Q}}$ is a state on $\mathcal{T}_{\mathbb{Q}}$. In fact $\sigma^Q(1 \otimes 1) = 1s(1) = 1$, and if $(q \otimes a) \odot (r \otimes b) = 0$, then $qa \odot rb = 0$. Then $\sigma^Q((q \otimes a) \oplus (r \otimes b)) = s^d(qa \oplus rb) = s^d(qa) \oplus s^d(rb) = \sigma^Q(q \otimes a) \oplus \sigma^Q(r \otimes b)$. Finally, σ^+ extends σ^Q by continuity:

$$\sigma^+(\alpha \otimes a) = \sup\left\{\sigma^Q(r \otimes a) : r \in \mathbb{Q}; r \le \alpha\right\},$$

and therefore the following hold:

- Clearly for each $\alpha \in [0,1]$ and $a \in M$, $\alpha \otimes a = \sup\{r \otimes a \mid r \in \mathbb{Q}, r \le \alpha\}$. In fact for each fixed $a \in M$, the partial function $\otimes^a_{\mathbb{Q}} : q \mapsto q \otimes a = qa$ is a continuos map over $\mathbb{Q}$. Therefore for each $\alpha \in [0,1]$ and $a \in M$, $\alpha \otimes a = \sup\{r \in \mathbb{Q} \mid r \le \alpha\} \otimes a = \sup\{r \otimes a \mid r \in \mathbb{Q}, r \le \alpha\}$.
- σ^+ is well-defined. In fact if $\alpha \otimes a = \beta \otimes b$, then $\sup\{r \otimes a \mid r \le \alpha\} = \sup\{q \otimes b \mid q \le \beta\}$. Now, for any fixed $x \in A$, $\sigma^Q(\cdot \otimes x) : t \mapsto ts(x)$ is a continuous map over $\mathbb{Q}$. Therefore $\sigma^+(\alpha \otimes a) = \sup\{\sigma^Q(r \otimes a) \mid r \le \alpha\} = \{\sigma^Q(q \otimes b) \mid q \le \beta\} = \sigma^+(\beta \otimes b)$. Hence σ^+ is well-defined.
- σ^+ is a state on $\mathcal{T}$. In fact $\sigma^+(1,1) = \sigma^Q(1,1) = 1$. Moreover if $(\alpha \otimes a) \odot (\beta \otimes b) = 0$, then from the continuity of $\odot$, one has $\sup\{(r \otimes a) \odot (q \otimes b) \mid r \le \alpha, b \le \beta\} = 0$. Therefore $\sigma^Q((r \otimes a) \oplus (q \otimes b)) = \sigma^Q(r \otimes a) \oplus \sigma^Q(q \otimes b)$. Thus $\sigma^+((\alpha \otimes a) \oplus (\beta \otimes b)) = \sup\{\sigma^Q((r \otimes a) \oplus (q \otimes b)) \mid r \le \alpha, q \le \beta\} = \sup\{\sigma^Q(r \otimes a) \oplus \sigma^Q(q \otimes b) \mid r \le \alpha, q \le \beta\} = \sigma^+(\alpha \otimes a) \oplus \sigma^+(\beta \otimes b)$ given the continuity of $\oplus$.

Therefore (1), (2) and (3) of Definition 3.1 hold. Moreover if $\alpha \in [0,1]$, then $\sigma^+(\alpha) = \alpha s(1) = \alpha$. Since for $x, y \in T$, $\sigma^+(x) \oplus \sigma^+(y) \in [0,1]$, also (4) is verified. Thus $(\mathcal{T}, \sigma^+)$ is a state MV-algebra. ■

6 The coherence problem

Given a finite probabilistic assessment $\Pr(E_1) = \alpha_1$, $\ldots$, $\Pr(E_n) = \alpha_n$, where $E_1, \ldots, E_n$ are fuzzy events, identified with sentences of Łukasiewicz logic, and $\alpha_1, \ldots, \alpha_n$ are real numbers in $[0,1]$, we consider the following problem: *Is there a state s on the Lindenbaum sentence algebra $L_Ł$ of Łukasiewicz logic such that for $i = 1, \ldots, n$, $s([E_i]) = \alpha_i$,* (where $[E_i]$ denotes the equivalence class of E_i modulo provable equivalence)*?* If such a state exist, then we say that the assessment is *coherent*. We want to find an equational characterization of the coherence problem when $\alpha_1, \ldots, \alpha_n$ are rational numbers. To this purpose, note that since Łukasiewicz logic is algebraizable in the sense of Blok and Pigozzi (cf [1]), the formulas $E_1, \ldots, E_n$ can be regarded as terms of MV-algebras as well. Now suppose that the α_i are rational numbers, say $\alpha_i = \frac{n_i}{m_i}$. Let $x_1, \ldots, x_n$ be fresh variables, and consider for $i = 1, \ldots, n$ the formulas the equations $\varepsilon_i : (m_i - 1)x_i = \neg x_i$. Moreover, for $i = 1, \ldots, n$, we denote by δ_i the equation $\sigma(E_i) = n_i x_i$. Then we can prove:

Theorem 6.1 *Let χ be the assessment consisting of the conditions $\Pr(E_i) = \frac{n_i}{m_i}$, $i = 1, \ldots, n$. The following are equivalent:*
(a) χ is coherent.
(b) The equations ε_i and δ_i, $i = 1, \ldots, n$, are satisfiable in some non-trivial state MV-algebra.

Proof. (a) $\Rightarrow$ (b). Given a state s on $L_Ł$ we can construct the state MV-algebra $(\mathcal{T}, \sigma^+)$ with $\mathcal{T} = [0,1]_{MV} \otimes L_Ł$ as in Theorem 5.2, thus getting a state MV-algebra whose operator σ^+ extends s. Let v be an evaluation on $(\mathcal{T}, \sigma^+)$ such that $v(x_i) = \frac{1}{m_i}$ (note that $\frac{1}{m_i} = \frac{1}{m_i} \otimes 1 \in \mathcal{T}$). Then equations ε_i are satisfied by v. Moreover, $\sigma^+(E_i) = \sigma^+(1 \otimes E_i) = 1s(E_i) = s(E_i) = \frac{n_i}{m_i} = n_i v(x_i)$, and also equations δ_i are satisfied by v.

(b) $\Rightarrow$ (a). Let $(\mathcal{M}, \sigma)$ be a state MV-algebra and v be an evaluation on $(\mathcal{M}, \sigma)$ satisfying equations ε_i and δ_i, $i = 1, \ldots, n$. Without loss of generality, we may assume that $\mathcal{M}$ is countably (or even finitely) generated, therefore there is an epimorphism h of $L_Ł$ onto $\mathcal{M}$ such that $h([x]) = v([x])$ for every propositional variable x. Thus $(m_i - 1)h([x_i]) = \neg h([x_i])$, and $\sigma(h([E_i])) = n_i h([x_i])$. Let I be a maximal MV-ideal of $\sigma(M)$, and define for every $[E] \in L_Ł$, $s^*([E]) = \sigma(h(E))/I$. Since quotients preserve identities, we have that $s^*([E_i]) = (n_i h([x_i]))/I$ and $(m_i - 1)(h([x_i])/I) = \neg(h([x_i])/I)$. Thus the canonical embedding of $\sigma(M)/I$ into $[0,1]_{MV}$ maps $h([x_i])/I$ into $\frac{1}{m_i}$ and $s^*([E_i])$ into $\frac{n_i}{m_i}$.

Finally, we verify that (modulo the canonical embedding of $\sigma(M)/I$ into $[0,1]_{MV}$), s^* is a state. That $s^*(1) = 1$ is clear. Moreover if $[E] \odot [F] = 0$ in $L_Ł$, then $h([E]) \odot h([F]) = 0$ holds in A, therefore $s^*([E]) \odot s^*([F]) = (\sigma(h([E])) \odot \sigma(h([F]))) / I = 0/I = 0$, and

$$s^*([E] \oplus [F]) = (\sigma(h([E]) \oplus h([F])))/I =$$
$$(\sigma(h([E])))/I \oplus (\sigma(h([F])))/I =$$
$$s^*([E]) \oplus s^*([F]).$$

Thus s^* is a state on $L_Ł$ satisfying the assessment χ, and the proof is finished. ∎

7 Conclusions and open problems

We introduced a unary operator σ on MV-algebras as to deal with the notion of *states* on MV-algebras and so defining the class of state MV-algebras. The main result of this paper says us that there is a canonical way of associating to each state MV-algebra $(\mathcal{M}, \sigma)$ a state on $\mathcal{M}$, and vice-versa, given a state s on an MV-algebra $\mathcal{M}$, one can define a state MV-algebra whose MV-reduct is obtained by tensor product of the standard MV-algebra $[0,1]_{MV}$ with $\mathcal{M}$.

As we remarked at the end of Section 4, the variety of state MV-algebras is not generated by its linearly-ordered members. For this reason, unlike the most common case, it is not possible to prove that the variety of state MV-algebras is generated by those algebras whose MV-reduct is the standard MV-algebra. In our opinion the most appropriate candidates to be the standard state MV-algebras are those algebras defined as in Section 5, and hence those algebras in the form $(\mathcal{T}, \sigma^+)$, $\mathcal{T}$ being the tensor product $[0,1]_{MV} \otimes \mathcal{M}$ and $\mathcal{M}$ being any (not necessarily linearly ordered) MV-algebra. Unfortunately we was not able, since now, to prove that an equation $\varphi \approx 0$, written in the language of state MV-algebras, holds in every state MV-algebra iff $\varphi \approx 0$ holds in some $(\mathcal{T}, \sigma^+)$. Therefore it can be reasonably considered as an open problem.

Another open problem is that one of studying the computational complexity for state MV-algebras, and applying it to the problem of deciding the coherence of rational assessments by the equational characterization we proved in Section 6.

Acknowledgement

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