Free MV-algebras

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Abstract

We describe n-generated free MV-algebras as MV-algebras having the lattice reduct which is a direct limit in the category of distributive lattices.

Keywords: MV-algebras, Lattices.

1 Preliminaries

We recall that an algebra $A = (A; 0, 1, \oplus, \cdot, \sim)$, [1], is said to be an MV -algebra iff it satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $\sim 0 = 1$;
6. $\sim 1 = 0$;
7. $x \cdot y = \sim(\sim x \oplus \sim y)$;
8. $\sim(\sim x \oplus y) \oplus y = \sim(\sim y \oplus x) \oplus x$.

Henceforth we shall write ab for $a \cdot b$ and a^n for $\underbrace{a \cdot \cdots \cdot a}_{n \text{ times}}$, for given $a, b \in A$. Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } \sim x \oplus y = 1.$$

$(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$xy \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

The unit interval of real numbers $[0, 1]$ endowed with the following operations: $x \oplus y = \min(1, x + y), x \cdot y = \max(0, x + y - 1), \sim x = 1 - x$, becomes an MV-algebra. Let Q denote the set of rational numbers, for $(0 \neq) m \in \omega$ we set $S_m = (S_m; \oplus, \cdot, \sim, 0, 1)$, where

$$S_m = \left\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\right\}.$$

Recall that an algebra $A \in \mathbf{K}$ is said to be a *free algebra* in a variety $\mathbf{K}$, if there exists a set $A_0 \subset A$ such that A_0 generates A and every mapping f from A_0 to any algebra $B \in \mathbf{K}$ is extended to a homomorphism h from A to B. In this case A_0 is said to be *the set of free generators* of A. If the set of free generators is finite then A is said to be a *finitely generated free algebra* .

2 Representation of n-generated MV-algebras

Let us denote by DL the category, and the variety as well, of distributive lattices, named by DL-algebras in the sequel, and lattice homomorphisms.

Notice that any MV-algebra is bounded distributive lattice and the lattice operations $\vee$ and $\wedge$ are definable by means of MV-algebra operations $\oplus$, $\cdot$ and $\sim$. So, any MV-algebra is a DL-algebra with respect to the operations $\vee, \wedge$.

Let $(A, \oplus, \cdot, \sim, 1, 0)$ be an MV-algebra. Define a functor $\mathcal{D}$ transforming MV-algebra $(A, \oplus, \cdot, \sim, 1, 0)$ into DL-algebra in the following way:

$$\mathcal{D}((A, \oplus, \cdot, \sim, 1, 0)) = (A, \vee, \wedge).$$

Then we have functor from the category of MV-algebras MV to the category DL.

Let $F_{V_n}(m)$ be the m-generated free MV-algebra in the variety $V_n = \mathcal{V}(\{S_1, ..., S_n\})$. We can consider any MV-algebra as (unbounded) distributive lattice (with lattice operations $\vee$ and $\wedge$). It is clear that $F_{V_n}(m)$ is embedded into $F_{V_{n+1}}(m)$ as MV-algebra and, consequently, as a lattice.

Let $g_1^{(n)}, ..., g_m^{(n)} \in F_{V_n}(m)$ be free generators of $F_{V_n}(m)$.

Let us define the injective mapping

$$\varepsilon_{n(n+1)} : \mathcal{D}(F_{V_n}(m)) \to \mathcal{D}(F_{V_{n+1}}(m))$$

for any $n \in \omega/\{0\}$ in the following way:

$$g_1^{(n)} \mapsto g_1^{(n+1)}, ..., g_m^{(n)} \mapsto g_m^{(n+1)} \text{ for any } n \in \omega/\{0\}.$$

It is clear that in the case when $n = 1$, $g_1^{(1)}, ..., g_m^{(1)}$ generate $F_{V_1}(m)$ $(= D_1^{(m)} = \mathcal{D}(F_{V_1}(m)))$ using only operations $\vee, \wedge$ and $\sim$. Then $\varepsilon_{12}(g_i^{(1)}) = g_i^{(2)}$, for $i = 1, ..., m$, and generate the algebra $D_2^{(m)}$ inside $F_{V_2}(m)$ by means of $g_1^{(2)}, ..., g_m^{(2)}$ using only operations $\vee, \wedge$ and $\sim$. It easy to check that $D_2^{(m)}$ is isomorphic to $D_1^{(m)} (= \mathcal{D}(F_{V_1}(m)))$. $\varepsilon_{12}(x \vee y) = \varepsilon_{12}(x) \vee \varepsilon_{12}(y)$, $\varepsilon_{12}(x \wedge y) = \varepsilon_{12}(x) \wedge \varepsilon_{12}(y)$.

In particular, if $m = 1$, then $\varepsilon_{12}(1) = g_1^{(2)} \vee \sim g_1^{(2)}$ $(\neq 1)$ and $\varepsilon_{12}(0) = g_1^{(2)} \wedge \sim g_1^{(2)}$ $(\neq 0)$. Observe that $\varepsilon_{12}(\mathcal{D}(F_{V_1}(1)))$ generates $F_{V_1}(2)$ using MV-algebra polynomials containing operations $\vee, \wedge, \sim$ and MV-algebra operation $\oplus$ (or $\cdot$) only one time. It is enough, since in V_2 holds the axiom $2x = 3x$ (and $2x$ is a Boolean element). Indeed, by means of the generator $g_1^{(2)}$ using $\vee, \wedge, \sim$ and polynomials containing operation $\oplus$ (or $\cdot$) only one time, we obtain elements $(0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ and, hence, any element of $F_{V_1}(2)$ by means of $\vee, \wedge$ and $g_1^{(2)}, \sim g_1^{(2)}$. Notice, that the embedding ε_{12} is not MV-algebra embedding, it is DL-algebra embedding. So, we define a functor $\mathcal{L}_2$ transforming DL-algebra $\mathcal{D}(F_{V_2}(1))$ into MV-algebra $F_{V_2}(1)$. Further, $\varepsilon_{23}(g_1^{(2)}) = g_1^{(3)}$ and close $\{g_1^{(3)}\}$ by means of operations $\vee, \wedge, \sim$. Then we obtain DL-algebra $D_3^{(1)}$ which is isomorphic to $\mathcal{D}(F_{V_1}(1))$. Analogically, as in the previous case, close $D_3^{(1)}$ using $\vee, \wedge, \sim$ and polynomials containing operation $\oplus$ (or $\cdot$) only one time. Then we obtain DL-algebra $\varepsilon_{23}(\mathcal{D}(F_{V_1}(2)))$ which is isomorphic to $\mathcal{D}(F_{V_1}(2))$. Further, close $\varepsilon_{12}(\mathcal{D}(F_{V_1}(2)))$ using $\vee, \wedge, \sim$ and polynomials containing operation $\oplus$ (or $\cdot$) only not more than on two places. Then we obtain the algebra $\mathcal{D}(F_{V_1}(3))$. So, we can define a functor $\mathcal{L}_3$ transforming DL-algebra $\mathcal{D}(F_{V_3}(1))$ into MV-algebra $F_{V_3}(1)$. And so on.

Analogically it is shown in general case when $m > 1$.

Further, $\varepsilon_{12}(g_i^{(2)}) = g_i^{(3)}$, for $i = 1, ..., m$, and generate the algebra $D_3^{(m)}$ inside $F_{V_3}(m)$ by means of $g_1^{(3)}, ..., g_m^{(3)}$ using only operations $\vee, \wedge$ and $\sim$. It easy to check that $D_3^{(m)}$ is isomorphic to $F_{V_1}(m)$ (as a lattice with involution). Moreover, $D_3^{(m)}$ generates $F_{V_3}(m)$, if we use the MV-algebra operations $\sim$ and $\oplus$ only two times. In other words, for generating $F_{V_3}(m)$ we use polynomials containing the operation $\sim$ and $\oplus$ only in not more than two places. It is enough, since in V_3 holds the axiom $3x = 4x$.

Let us consider the direct system $(\mathcal{D}(F_{V_n}(m)), \varepsilon_{n(n+1)})_{n \in \omega/\{0\}}$. Let $\mathcal{D}(F(m))$ be the direct limit of the direct system $(\mathcal{D}(F_{V_n}(m)), \varepsilon_{n(n+1)})_{n \in \omega/\{0\}}$. The elements of $\mathcal{D}(F(m))$ are infinite sequences $(a_i, a_{i+1}, a_{i+2}, ...)$ for $i \in \omega$ such that $a_i \in \mathcal{D}(F_{V_i}(m))$ and a_i has no inverse image with respect to $\varepsilon_{(i-1)i}$, and, moreover, $\varepsilon_{i_1(i_1+1)}(a_{i_1}) = a_{i_1+1}$ for any $i_1 \geq i$. Now we define the operation $\oplus$ on $\mathcal{D}(F(m))$ in the following way.

Let $a = (a_i, a_{i+1}, a_{i+2}, ...), b = (b_j, b_{j+1}, b_{j+2}, ...)$ and $a, b \in F(m)$, and $i \leq j$. It is clear that a_i (respectively, b_j) represented by a valuation of an MV-algebra polynomial containing operation $\oplus$ only on $i - 1$ (respectively, $j - 1$) places. Then we consider elements $a_{max(i,j)}$ and $b_{max(i,j)}$. It is clear that they belong to $\mathcal{D}(F_{V_{max(i,j)}}(m))$. Choose the least number $k \geq max(i, j)$ such that $a_{max(i,j)} + b_{max(i,j)} \in \mathcal{D}(F_{V_k}(m))$. Then $a \oplus b = (a_k + b_k, a_{k+1} + b_{k+1}, a_{k+2} + b_{k+2}, ...)$. Then we convert $\mathcal{D}(F(m))$ into MV-algebra which we denote by $F_{MV}(m)$.

Theorem 2.1. *The MV-algebra $F_{MV}(m)$ is m-generated free MV-algebra.*

References

[1] C. C. Chang, Algebraic Analysis of Many-Valued Logics, Trans. Amer. Math. Soc. 88 (1958) 467-490.