Using quasiarithmetic means in a sequential decision procedure

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Abstract

We study in this paper the following sequential decision procedure. First, the members of a group show their opinions on all the members, regarding a specific attribute. Taking into account this information, quasiarithmetic means and a family of thresholds, a subgroup of individuals is selected: those members whose collective assessments (obtained through a quasiarithmetic mean) reach a specific threshold. After that, only the assessments of this qualified subgroup are taken into account emerging a new subgroup with the aggregation phase. We analyze when this recursive procedure converges providing a final subgroup of qualified members.

Keywords: Group decision making, Aggregation operators, Quasiarithmetic means, Qualification.

1 Introduction

Consider a group of experts that has to decide which of its members should participate in a concrete task or constituting a committee. In other fields of research there are works where the problem arises in choosing the members of the society satisfying a social identity (see Kasher and Rubinstein [6]), with respect to a general attribute (see Samet and Schmeidler [10]) or, in a very different context, which elements of a society should be considered to delimitate a social graph (see Laumann, Marsden and Prensky [8]). In Ballester and García-Lapresta [1] a recursive procedure for selecting a final group of individuals is introduced.

We examine in this paper the use of that sequential procedure where in each stage a subgroup is selected, and taking the opinions of this subgroup a new subgroup emerges. This tries to reflect the idea that selected members have more valuable opinions. For doing the selection, an aggregation operator and a threshold are considered over the gradual opinions.

Aggregation operators allow us to generate a collective assessment to each individual of the group taking into account the individual opinions (see Fodor and Roubens [4], Grabisch, Orlovski and Yager [5] and Calvo, Kolesárova, Komorníková and Mesiar [3], among others).

Among the large variety of aggregation operators that we can use for our sequential procedure, the case of OWA ("Ordered Weighted Averaging") operators (Yager [11]) have been widely discussed in Ballester and García-Lapresta [1]. In the present paper, we complement that study by considering a different family of operators that has attracted significant attention in the literature, the class of quasiarithmetic means.

Quasiarithmetic means were characterized by Kolmogoroff [7] and Nagumo [9]. In Bullen, Mitrinović and Vasić [2] there is an exhaustive study on means (chapters IV and VI are devoted to quasiarithmetic means). See Fodor and Roubens [4, 5.5] and Calvo, Kolesárova, Komorníková and Mesiar [3, 4.3] as well.

The paper is organized as follows. In Section 2 we consider again the basic framework for developing the group decision procedure. Section 3 briefly describes the sequential procedure subject of analysis. Section 4 includes some important families of collective evaluation mechanisms suitable for the recursive decision problem. In Section 5 we announce and describe the results of the paper. Finally, Section 6 contains some conclusions.

2 Preliminaries

Consider a finite set of individuals $N = \{1, 2, \dots, n\}$ with $n \geq 2$. We use 2^N to denote the power set of N, i.e., the set of all the subsets of N, and $|S|$ is the cardinal of S.

A *profile* is an $n \times n$ matrix

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i1} & \cdots & p_{ij} & \cdots & p_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{nj} & \cdots & p_{nn} \end{pmatrix}$$

with values in the unit interval, where p_{ij} is the assessment with which the individual i evaluates individual j as being qualified to belong to the committee in question. The set of profiles is denoted by $\mathcal{P}$. Given a subset of individuals $S \subseteq N$, P_S denotes the $|S| \times n$ submatrix of P composed by those $i-$rows with $i \in S$. Given $j \in N$, we denote by P_S^j the j-th column vector of P_S.

Definition 2.1 *A* Committees' Evaluation Mechanism (CEM) *is a family of functions* $\{v_S\}$*, with* $\emptyset \neq S \subseteq N$*, where* $v_S : \mathcal{P} \times N \longrightarrow [0,1]$ *is a function that, given* $P \in \mathcal{P}$*, assigns a collective assessment* $v_S(P,j) \in [0,1]$ *to each individual* $j \in N$ *in such a way that* $v_S(P,j) = v_S(Q,j)$ *for all* $P, Q \in \mathcal{P}$ *satisfying* $P_S^j = Q_S^j$*.*

Implicit in the previous definition is the fact that the collective assessment that the subgroup S provides to individual j, $v_S(P,j)$, only depends on the individual assessments of S on the individual j.

A very natural way to transform the gradual opinion that the CEM provides into a dichotomic assessment is by means of thresholds or quotas. Intuitively, an individual is going to be qualified if the collective assessment is above a fixed quota.

Definition 2.2 *A family of values* $\{\alpha_S\}$, $\emptyset \neq S \subseteq N$, *and* $\alpha_S \in (0,1]$ *for every* S*, is called a* Threshold Mechanism (TM)*.*

Given a CEM $\{v_S\}$ *and a TM* $\{\alpha_S\}$*, the family of functions* $\{V_S\}$*, with* $S \subseteq N$*, where* $V_S : \mathcal{P} \longrightarrow 2^N$ *is the function that, given* $P \in \mathcal{P}$*, qualifies a new subgroup* $V_S(P)$ *as follows:*

$$V_S(P) = \begin{cases} \{j \in N \mid v_S(P,j) \geq \alpha_S\}, & \text{if } S \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The family of functions $\{V_S\}$ *is called the* Committees' Qualification Mechanism (CQM) *associated with* $\{\alpha_S\}$ *and* $\{v_S\}$*.*

Definition 2.3 *Given a CQM* $\{V_S\}$*, the sequence* $\{S_t\}$*, where* $S_1 = N$ *and* $S_{t+1} = V_{S_t}(P)$*, is called* committees' sequence*. A committees' sequence is said to be* convergent *if* $\{S_t\}$ *has a limit* $\lim S_t$ *(and it is also said that the sequence* converges *to* $\lim S_t$*). The CQM is said to be* convergent *if for any* $P \in \mathcal{P}$ *the committees' sequence generated is convergent.*

As noted in Ballester and García-Lapresta [1], since the society is a finite set, chain convergence can also be expressed as: there exists a positive integer q such that $S_t = S_q$ for every $t \geq q$.

3 An overview of the sequential procedure

We present a brief description of the sequential procedure introduced in Ballester and García-Lapresta [1]. Consider a group of individuals that have to decide which of its members are adequate for a particular task or possess an attribute or ability. The sequential procedure makes use of their opinions (the profile matrix), the social ways to determine qualification (CEM and CQM) in a sequential way (committees' sequence). The aim is to determine a final set of qualified individuals (reason for which convergence is fundamental).

1. The society endows itself of a mechanism of decision, a CQM.

2. All the opinions are stated and the profile matrix is obtained.

3. The whole society (S_1), determines a initial set of qualified members, S_2, by using its associated function V_N in the CQM.

4. The members in S_2 have now special relevance, as they are roughly considered the set of qualified members. They determine, by using the function V_{S_2} in the CQM, who should be the set of qualified members, S_3.

5. The process is iterated until a limit set (or a cycle) is attained. If the limit set is obtained, this is the set of qualified members for the given profile of opinions.

The problem is to determine mechanisms that, given any possible social opinion, determine a final set of qualified individuals (convergence). Notice that the mechanism with which a society endows itself may be useful for a large variety of situations and problems and therefore, it has to be flexible enough to answer positively to any plausible matrix of individual opinions.

4 Describing families of CEMs

In this section, we focus our attention on how general aggregation operators and classes of quasiarithmetic means can be used in the recursive procedure introduced in the previous sections.

4.1 The use of aggregation operators to construct CEMs

Typically, aggregation operators are defined for a given number of individual opinions. Notice however that for our model to apply, we have to consider collections of operators, one for each possible subgroup of individuals in society. Although the ways to afford this task are unlimited, we reconsider in this paper the intuitive two assumptions done in Ballester and García-Lapresta [1].

SE *Self-Exclusion*: When an individual had to decide on herself, she will exclude her opinion when possible (i.e., she is not the only reviewer).

CS *Cardinality-Symmetry*: When the same number of opinions have to be aggregated, the same operator will be applied (independently of the names of the reviewers).

Both of them could be eliminated without altering significatively the results, but we consider this model more natural.

4.2 Quasiarithmetic means

Definition 4.1 *Let* $\varphi : [0,1] \longrightarrow [0,1]$ *be an* order automorphism*, i.e.,* φ *is bijective and increasing. The* quasiarithmetic mean of dimension p associated with φ *is the function* $f^p_\varphi : [0,1]^p \longrightarrow [0,1]$ *defined by*

$$f^p_\varphi(a_1,\dots,a_p) = \varphi^{-1}\left(\frac{\varphi(a_1)+\cdots+\varphi(a_p)}{p}\right).$$

It is natural to consider quasiarithmetic means that use the same order automorphism φ across dimensions. The construction of the associated CEM function $\{v_S\}$, therefore, would be, for any $\emptyset \neq S \subseteq N$:

1. The quasiarithmetic mean of dimension $|S|$, $f^{|S|}_\varphi$, applied to P^j_S, if $j \notin S$.
2. The quasiarithmetic mean of dimension $|S|-1$, $f^{|S|-1}_\varphi$, applied to $P^j_{S\setminus\{j\}}$, if $j \in S \neq \{j\}$.
3. The quasiarithmetic mean of dimension 1, f^1_φ, i.e., the identity, applied to $P^j_{\{j\}} = p_{jj}$, if $S = \{j\}$.

Notice that the average aggregator is indeed a quasiarithmetic mean simply considering the identity automorphism.

Example 4.2 The *root-power* quasiarithmetic means can be defined through the order automorphism $\varphi(x) = x^q$, with q a positive integer. The associated CEM $\{r_S\}$ is described in Table 1.

Table 1: CEM associated with the *root-power* quasiarithmetic mean

Cases	CEM: $r_S(P,j)$
1. $j \notin S$	$\left(\frac{1}{\|S\|}\sum_{i\in S} p^q_{ij}\right)^{\frac{1}{q}}$
2. $j \in S \neq \{j\}$	$\left(\frac{1}{\|S\|-1}\sum_{i\in S\setminus\{j\}} p^q_{ij}\right)^{\frac{1}{q}}$
3. $S = \{j\}$	p_{jj}

The associated CQM $\{R_S\}$ is defined by

$$j \in R_S(P) \;\Leftrightarrow\; r_S(P,j) \geq \alpha_S.$$

Example 4.3 The *exponential* quasiarithmetic means can be defined through the order automorphism

$$\varphi(x) = \frac{e^{\beta x}-1}{e^\beta - 1}$$

with $\beta \neq 0$. The associated CEM $\{e_S\}$ is described in Table 2.

Table 2: CEM associated with the *exponential* quasiarithmetic mean

Cases	CEM: $e_S(P,j)$
1. $j \notin S$	$\frac{1}{\beta}\ln\frac{\sum_{i\in S} e^{\beta p_{ij}}}{\|S\|}$
2. $j \in S \neq \{j\}$	$\frac{1}{\beta}\ln\frac{\sum_{i\in S\setminus\{j\}} e^{\beta p_{ij}}}{\|S\|-1}$
3. $S = \{j\}$	p_{jj}

The associated CQM $\{E_S\}$ is defined by

$$j \in E_S(P) \;\Leftrightarrow\; e_S(P,j) \geq \alpha_S.$$

5 The results

In this section we aim to provide some sufficient conditions for the convergence of CQMs associated with quasiarithmetic means.

Definition 5.1 *Given an order automorphism $\varphi : [0,1] \longrightarrow [0,1]$, the function $g_\varphi : (0,1] \longrightarrow (0,1]$ is defined by*

$$g_\varphi(x) = \frac{\varphi(x)}{x}.$$

Lemma 5.2 *Let φ be an order automorphism. If the function g_φ is increasing, then $x\,\varphi(y) \geq \varphi(x\,y)$ for all $x, y \in [0,1]$.*

PROOF: First of all, notice that the result is true if $x = 0$ or $y = 0$, and $x\,y \leq y$ for all $x, y \in [0,1]$. Since g_φ is increasing, we have

$$\frac{\varphi(y)}{y} = g_\varphi(y) \geq g_\varphi(x\,y) = \frac{\varphi(x\,y)}{x\,y}$$

for all $x, y \in (0,1]$, equivalently, $\varphi(y) \geq \dfrac{\varphi(x\,y)}{x}$ and $x\,\varphi(y) \geq \varphi(x\,y)$. ∎

Proposition 5.3 *Let $\{V_S\}$ be a CQM associated with a quasiarithmetic mean $\{f^p_\varphi\}$ and a TM $\{\alpha_S\}$. If the following two conditions are satisfied:*

1. *The function g_φ is increasing,*
2. *For all non-empty sets $U, V \subseteq N$ such that $U \subseteq V$ it holds $|U|\,\alpha_U \geq (|V|-1)\,\alpha_V$,*

then $\{V_S\}$ is convergent.

PROOF: Suppose conditions 1 and 2 hold. Given any profile P, we prove by induction that the committee sequence is decreasing, hence, convergent. Obviously, $S_2 \subseteq S_1 = N$. Suppose $S_{k+1} \subseteq S_k$ is true for any $k \in \{1, \ldots, t-1\}$ (in particular $S_t \subseteq S_{t-1}$). In order to prove $S_{t+1} \subseteq S_t$ by way of contradiction, suppose there exists an individual $j \in S_{t+1}$ such that $j \notin S_t$. Consider the greatest integer m such that $j \in S_m$, with $m \leq t-1$. Notice that this is well-defined, since $S_1 = N$. By the induction hypothesis, it must be that $S_t \subseteq S_m$ and, consequently, $S_t = S_t \setminus \{j\} \subseteq S_m \setminus \{j\}$. Therefore,

$$\frac{\displaystyle\sum_{i\in S_m\setminus\{j\}} \varphi(p_{ij})}{|S_m|-1} \geq \frac{\displaystyle\sum_{i\in S_t} \varphi(p_{ij})}{|S_m|-1}.$$

Since φ^{-1} is also increasing, we have

$$\varphi^{-1}\left(\frac{\displaystyle\sum_{i\in S_m\setminus\{j\}} \varphi(p_{ij})}{|S_m|-1}\right) \geq \varphi^{-1}\left(\frac{\displaystyle\sum_{i\in S_t} \varphi(p_{ij})}{|S_m|-1}\right) =$$

$$= \varphi^{-1}\left(\frac{|S_t|}{|S_m|-1}\,\frac{\displaystyle\sum_{i\in S_t} \varphi(p_{ij})}{|S_t|}\right).$$

By the fact that $j \in S_{t+1} = V_{S_t}(P)$ and $j \notin S_t$, we know that

$$\varphi^{-1}\left(\frac{\displaystyle\sum_{i\in S_t} \varphi(p_{ij})}{|S_t|}\right) \geq \alpha_{S_t}.$$

Being φ increasing, this is equivalent to

$$\frac{\displaystyle\sum_{i\in S_t} \varphi(p_{ij})}{|S_t|} \geq \varphi(\alpha_{S_t}).$$

Hence, substituting:

$$\varphi^{-1}\left(\frac{|S_t|}{|S_m|-1}\,\frac{\displaystyle\sum_{i\in S_t} \varphi(p_{ij})}{|S_t|}\right) \geq$$

$$\geq \varphi^{-1}\left(\frac{|S_t|}{|S_m|-1}\,\varphi(\alpha_{S_t})\right).$$

Lemma 5.2 and the fact that φ^{-1} is increasing guarantee that

$$\varphi^{-1}\left(\frac{|S_t|}{|S_m|-1}\,\varphi(\alpha_{S_t})\right) \geq \varphi^{-1}\left(\varphi\left(\frac{|S_t|}{|S_m|-1}\,\alpha_{S_t}\right)\right).$$

Condition 2 ensures $|S_t|\,\alpha_{S_t} \geq (|S_m|-1)\,\alpha_{S_m}$. Then,

$$\varphi^{-1}\left(\varphi\left(\frac{|S_t|}{|S_m|-1}\,\alpha_{S_t}\right)\right) \geq \varphi^{-1}(\varphi(\alpha_{S_m})) = \alpha_{S_m}.$$

Hence we obtain

$$\varphi^{-1}\left(\frac{\displaystyle\sum_{i\in S_m\setminus\{j\}} \varphi(p_{ij})}{|S_m|-1}\right) \geq \alpha_{S_m}$$

or, equivalently, $j \in V_{S_m}(P) = S_{m+1}$, leading to an absurd and concluding the proof. ∎

Corollary 5.4 *If for all non-empty sets $U, V \subseteq N$ such that $U \subseteq V$ it holds $|U|\,\alpha_U \geq (|V|-1)\,\alpha_V$, then any associated CQM $\{R_S\}$ is convergent.*

PROOF: It is sufficient to take into account that the function g_φ, defined by

$$g_\varphi(x) = \frac{x^r}{x} = x^{r-1},$$

is increasing. ∎

Corollary 5.5 *Let $\{V_S\}$ be a CQM associated with a quasiarithmetic mean $\{f_\varphi^p\}$ and a TM $\{\alpha_S\}$. If φ is convex and for all non-empty sets $U, V \subseteq N$ such that $U \subseteq V$ it holds $|U|\,\alpha_U \geq (|V|-1)\,\alpha_V$, then $\{V_S\}$ is convergent.*

PROOF: We prove by way of contradiction that the function g_φ is increasing. Suppose that there exist $x, y \in (0,1]$ such that $x < y$ but $g_\varphi(x) > g_\varphi(y)$. In this case,

$$\frac{\varphi(x)}{x} > \frac{\varphi(y)}{y} \;\Rightarrow\; \varphi(x) > \frac{x}{y}\,\varphi(y).$$

Since $\varphi(0) = 0$ and $\dfrac{x}{y} \in (0,1)$, we have

$$\varphi\left(\frac{x}{y}\,y + \left(1 - \frac{x}{y}\right)0\right) = \varphi(x) > \frac{x}{y}\,\varphi(y) =$$

$$= \frac{x}{y}\,\varphi(y) + \left(1 - \frac{x}{y}\right)\varphi(0).$$

This fact contradicts the hypothesis that φ was convex. Thus, the function g_φ is increasing and the associated CQM $\{V_S\}$ is convergent. ■

The previous corollary gives us a very simple result about the convergence of exponential quasiarithmetic means.

Corollary 5.6 *If for all non-empty sets $U, V \subseteq N$ such that $U \subseteq V$ it holds $|U|\,\alpha_U \geq (|V|-1)\,\alpha_V$, and $\beta > 0$, then the associated CQM $\{E_S\}$ is convergent.*

PROOF: Given Corollary 5.5, the fact that the function

$$\varphi(x) = \frac{e^{\beta x} - 1}{e^{\beta} - 1}$$

is convex in $(0,1]$ for any $\beta > 0$, allows us to conclude as desired. ■

6 Concluding remarks

In this paper, we have analyzed the possibility of using different aggregation operators in a previously introduced sequential evaluation mechanism.

The main aim was to obtain results about the convergence of the associated committees' qualification mechanisms. We have provided some sufficient conditions for the convergence of mechanisms based on quasiarithmetic means, paying special attention to root-power and exponential means and those generated by convex order automorphisms.

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