

A characterization of the ambiguity and fuzziness by means of activity orders on the closed intervals in $[0,1]$

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Abstract

We take as a departure point the definition of activity orders, intending to measure the ambiguity and fuzziness of any closed interval in $[0, 1]$. To do it, we define a family of orders on the set of closed intervals of a distributive lattice L , that will allow us to set up some preorders associated to the ambiguity and fuzziness.

Keywords: activity orders, fuzziness and ambiguity.

1 Introduction. Activity orders

If $\mathcal{J}[L]$ represents the set of closed intervals in a distributive lattice L , then some order relations on $\mathcal{J}[L]$ are defined to set up preorders associated to the fuzziness and ambiguity.

To do it, we will start from the concept of *activity order* [10, 9, 11] that was introduced by Serra [10], in the context of the morphologic operators on images of \mathbb{R}^n or \mathbb{Z}^n , as follows.

1.1 Definition

Let $\Omega = \{\Psi/\Psi : \wp(\mathbb{R}^2) \rightarrow \wp(\mathbb{R}^2)\}$ be a set of functions that transforms the images A of the plane \mathbb{R}^2 with the order $(\Psi_1 \leq \Psi_2) \iff (\Psi_1(A) \subseteq \Psi_2(A), \forall A \in \mathbb{R}^2)$. Let $i \in \Omega$ be the identity operator: $i(A) = A, \forall A \in \mathbb{R}^2$. The activity order between operators Ψ_1, Ψ_2 is defined by:

$$\Psi_1 \preceq \Psi_2 \iff \begin{cases} (i \wedge \Psi_1) \geq (i \wedge \Psi_2) \\ \& \\ (i \vee \Psi_1) \leq (i \vee \Psi_2) \end{cases}$$

Later, Heijmans and Keshet [8] extended this definition to any lattice, generating a family of orders that are shown in the next:

1.2 Definition

Let (L, \leq) be a lattice and $\omega_0 \in L$. The binary relation \preceq_{ω_0} is defined on L by the expression:

$$(x \preceq_{\omega_0} y) \iff \begin{cases} (\omega_0 \wedge y) \leq (\omega_0 \wedge x) \\ \& \\ (\omega_0 \vee y) \geq (\omega_0 \vee x) \end{cases} \quad (1)$$

where \vee and \wedge represent the join and meet in L respectively.

1.3 Proposition

The relation \preceq_{ω_0} is reflexive and transitive for each $\omega_0 \in L$, that is:

$$\begin{aligned} &(\alpha \preceq_{\omega_0} \alpha), \forall \alpha \in L \\ &((\alpha \preceq_{\omega_0} \beta) \& (\beta \preceq_{\omega_0} \gamma)) \implies (\alpha \preceq_{\omega_0} \gamma). \end{aligned}$$

It is not necessarily antisymmetric, since the lattice in Figure 1 verifies: $\alpha \preceq_{\omega_0} \beta, \beta \preceq_{\omega_0} \alpha$ and $\alpha \neq \beta$. Consequently, the binary relation \preceq_{ω_0} , in general, is not an order relation.

An immediate consequence of the studied theory in [8] is the following result that tells us when the previous relation is an order.

1.4 Theorem

If the ordered set (L, \leq) is a distributive lattice (L, \vee, \wedge) , it is verified:

- (i) For all $\omega_0 \in L$, the relation \preceq_{ω_0} defined in (1) is an order relation on L .
- (ii) The partially ordered set (L, \preceq_{ω_0}) associated to ω_0 is a meet-semilattice $(L, \lambda_{\omega_0}, \mathbf{0}_{\omega_0})$ with a minimum element $\mathbf{0}_{\omega_0} = \omega_0$, and where the meet operation λ_{ω_0} is given, for all $(\alpha, \beta) \in L^2$, by

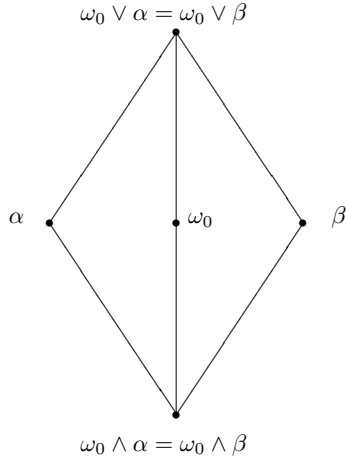


Figure 1: \preceq_{ω_0} is not an order

$$(\alpha \lambda_{\omega_0} \beta) = \xi, \text{ where}$$

$$\xi = [\omega_0 \wedge (\alpha \vee \beta)] \vee (\alpha \wedge \beta) = [\omega_0 \vee (\alpha \wedge \beta)] \wedge (\alpha \vee \beta).$$

Hence, we will work with the distributive lattices and with the following family of orders.

1.5 Definition

If (L, \leq) is a distributive lattice and $\omega_0 \in L$, then the order relation \preceq_{ω_0} defined in (1), is said to be the ω_0 -activity order.

1.6 Example

In Figure 2, we represent some examples of the family of orders $(\preceq_{\omega})_{\omega \in [0,1]}$ obtained in the chain $([0, 1], \leq)$.

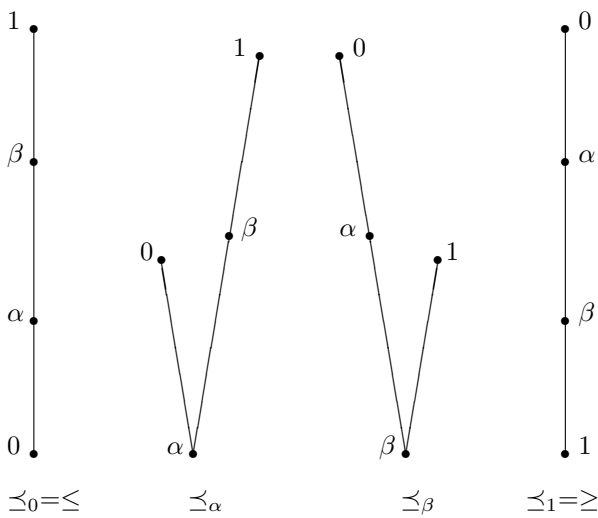


Figure 2: Orders of the family $(\preceq_{\omega})_{\omega \in [0,1]}$

Note that $\preceq_{\omega_0}^{op}$, the dual order of \preceq_{ω_0} , is given by:

$$(\alpha \preceq_{\omega_0}^{op} \beta) \iff \begin{cases} (\omega_0 \wedge \alpha) \leq (\omega_0 \wedge \beta) \\ \& \\ (\omega_0 \vee \alpha) \geq (\omega_0 \vee \beta) \end{cases}$$

Therefore, $(L, \preceq_{\omega_0}^{op})$ is a joint-semilattice where

$$(\alpha \vee_{\omega_0}^{op} \beta) = [\omega_0 \wedge (\alpha \vee \beta)] \vee (\alpha \wedge \beta) = [\omega_0 \vee (\alpha \wedge \beta)] \wedge (\alpha \vee \beta),$$

and ω_0 is the maximum element of $(L, \preceq_{\omega_0}^{op})$.

Note that, for every distributive lattice (L, \leq) with a minimum element $\mathbf{0}$, it is verified: $(\preceq_{\mathbf{0}}) = (\leq)$. If there is a maximum element $\mathbf{1}$, then the relation $\preceq_{\mathbf{1}}$ is \geq , (opposite to \leq).

Let us see a particular case of this order relation that has already been defined.

1.7 Proposition

The "sharpened" order \leq^s [5, 13] used in fuzziness degree measures [2] and in the theory of fuzzy entropies [12], is a particular case of the orders \preceq_{ω} . Specifically, for $L = [0, 1]$, $\omega = \frac{1}{2}$:

$$(\alpha \leq^s \beta) \iff (\alpha \preceq_{1/2}^{op} \beta) \iff \begin{cases} \min(\frac{1}{2}, \alpha) \leq \min(\frac{1}{2}, \beta) \\ \& \\ \max(\frac{1}{2}, \alpha) \geq \max(\frac{1}{2}, \beta) \end{cases}$$

If we consider now an interval $[\omega_0]$ represented by $[\omega_0] = [\underline{\omega}_0, \overline{\omega}_0] \in \mathcal{J}[L]$, with $\underline{\omega}_0 \leq \overline{\omega}_0$, we can characterize the family of orders that we have defined in the sets of intervals $(\mathcal{J}[L], \leq)$.

1.8 Proposition

If (L, \leq) is a distributive lattice and $(\mathcal{J}[L], \leq)$ is the distributive lattice of the closed intervals in L with the order

$$([\underline{\alpha}, \overline{\alpha}] \leq [\underline{\beta}, \overline{\beta}]) \iff (\underline{\alpha} \leq \underline{\beta}) \& (\overline{\alpha} \leq \overline{\beta}),$$

then

$$([\underline{\alpha}, \overline{\alpha}] \preceq_{[\omega_0]} [\underline{\beta}, \overline{\beta}]) \iff (\underline{\alpha} \preceq_{\underline{\omega}_0} \underline{\beta}) \& (\overline{\alpha} \preceq_{\overline{\omega}_0} \overline{\beta})$$

Proof Given the interval $[\omega_0] = [\underline{\omega}_0, \overline{\omega}_0]$, the ω_0 -activity order associated to the element ω_0 is

$$(\alpha \preceq_{\omega_0} \beta) \iff \begin{cases} (\omega_0 \wedge \alpha) \geq (\omega_0 \wedge \beta) \\ \& \\ (\omega_0 \vee \alpha) \leq (\omega_0 \vee \beta) \end{cases}$$

On the other hand, if we take the order relation defined on the set of intervals

$$([\alpha, \bar{\alpha}] \preceq_{\omega_0} [\beta, \bar{\beta}]) \iff \begin{cases} ([\underline{\omega}_0, \bar{\omega}_0] \wedge [\alpha, \bar{\alpha}]) \geq ([\underline{\omega}_0, \bar{\omega}_0] \wedge [\beta, \bar{\beta}]) \\ \& \\ ([\underline{\omega}_0, \bar{\omega}_0] \vee [\alpha, \bar{\alpha}]) \leq ([\underline{\omega}_0, \bar{\omega}_0] \vee [\beta, \bar{\beta}]) \end{cases}$$

Then, it follows that

$$([\alpha, \bar{\alpha}] \preceq_{\omega_0} [\beta, \bar{\beta}]) \iff \begin{cases} [\underline{\omega}_0 \wedge \alpha, \bar{\omega}_0 \wedge \bar{\alpha}] \geq [\underline{\omega}_0 \wedge \beta, \bar{\omega}_0 \wedge \bar{\beta}] \\ \& \\ [\underline{\omega}_0 \vee \alpha, \bar{\omega}_0 \vee \bar{\alpha}] \leq [\underline{\omega}_0 \vee \beta, \bar{\omega}_0 \vee \bar{\beta}] \end{cases} \iff \begin{cases} \underline{\omega}_0 \wedge \alpha \geq \underline{\omega}_0 \wedge \beta \\ \& \\ \bar{\omega}_0 \wedge \bar{\alpha} \geq \bar{\omega}_0 \wedge \bar{\beta} \\ \& \\ \underline{\omega}_0 \vee \alpha \leq \underline{\omega}_0 \vee \beta \\ \& \\ \bar{\omega}_0 \vee \bar{\alpha} \leq \bar{\omega}_0 \vee \bar{\beta} \end{cases} \iff \begin{cases} \alpha \preceq_{\omega_0} \beta \\ \& \\ \bar{\alpha} \preceq_{\bar{\omega}_0} \bar{\beta} \end{cases}$$

□

1.9 Example

If $L = [0, 1]$, then we obtain:

$$([\alpha] \preceq_{[0,1]} [\beta]) \iff (\alpha \preceq_0 \beta) \& (\bar{\alpha} \preceq_1 \bar{\beta}),$$

that is

$$([\alpha] \preceq_{[0,1]} [\beta]) \iff (\alpha \leq \beta) \& (\bar{\alpha} \geq \bar{\beta})$$

It proves that $\preceq_{[0,1]}$ is the inclusion order \supseteq between intervals:

$$([\alpha] \preceq_{[0,1]} [\beta]) \iff ([\alpha] \supseteq [\beta]).$$

Notation. Given a distributive lattice (L, \leq) , for every element $\omega_0 \in L$, the order relation dual of \preceq_{ω_0} will be denoted by \leq_{ω_0} . That is, $\leq_{\omega_0} = \preceq_{\omega_0}^{op}$.

Therefore, we will have that:

$$\alpha \leq_{\omega_0} \beta \iff \begin{cases} \omega_0 \wedge \alpha \leq \omega_0 \wedge \beta \\ \& \\ \omega_0 \vee \alpha \geq \omega_0 \vee \beta \end{cases}$$

As we said before, the set L with this order relation is a joint-semilattice where the joint of any two elements is given by

$$\alpha \vee_{\omega_0} \beta = \{\omega_0 \wedge (\alpha \vee \beta)\} \vee (\alpha \wedge \beta)$$

and where ω_0 is the maximum element.

1.10 Example

In particular, if $L = [0, 1]$ and $(\mathcal{J}[L], \leq)$ is the distributive lattice of the closed intervals in L with the order

$$[a, b] \leq [c, d] \iff a \leq c \ \& \ b \leq d$$

then

$$[a, b] \leq_{[1,1]} [c, d] \iff \begin{cases} [a, b] \wedge [1, 1] \leq [c, d] \wedge [1, 1] \\ \& \\ [a, b] \vee [1, 1] \geq [c, d] \vee [1, 1] \end{cases} \iff [a, b] \leq [c, d]$$

that is, the order $\leq_{[1,1]}$ is the usual order between intervals \leq .

2 Preorder relation associated to the fuzziness of the intervals

In the Fuzzy sets theory, diverse magnitudes have been defined in order to measure the fuzziness [6, 7]. Similar measures have also been considered for interval-valued fuzzy sets [3].

From a different point of view, using the order relations \leq_{ω_0} , we will introduce in this section some preorder relations on the set of closed intervals in $L = [0, 1]$, that later we will allow us to define other measures of fuzziness and ambiguity on the interval-valued fuzzy sets.

In order to set up comparisons among the fuzziness of the elements of the set of closed intervals in $L = [0, 1]$ we will remind the relation 'at least as fuzzy as' defined by De Luca and Termini in [6].

2.1 Definition

Let (L, \leq) be a distributive lattice endowed with a negation $'$. Given two elements $x, y \in L$, y is said to be *at least as fuzzy* as x , and we will represent by

$$x \prec_{fuz} y \iff \begin{cases} x \wedge x' \leq y \wedge y' \\ \& \\ x \vee x' \geq y \vee y' \end{cases}$$

The intuitive idea of this definition is that y and y' are closer in L than x and x' .

2.2 Proposition

The relation \prec_{fuz} is a preorder relation on L .

Proof The relation \prec_{fuz} is reflexive:

$$\left. \begin{array}{l} x \wedge x' \leq x \wedge x' \\ \& \\ x \vee x' \geq x \vee x' \end{array} \right\} \iff x \prec_{fuz} x, \ \forall x \in L$$

It is also transitive:

$$\left. \begin{array}{l} x \prec_{fuz} y \Rightarrow x \wedge x' \leq y \wedge y' \ \& \ x \vee x' \geq y \vee y' \\ y \prec_{fuz} z \Rightarrow y \wedge y' \leq z \wedge z' \ \& \ y \vee y' \geq z \vee z' \end{array} \right\} \Rightarrow x \prec_{fuz} z$$

Therefore, the relation \prec_{fuz} is a preorder relation. \square

Remark. The previous relation is not an order relation because it is not an antisymmetrical relation:

$$\left. \begin{array}{l} x \prec_{fuz} y \Rightarrow x \wedge x' \leq y \wedge y' \ \& \ x \vee x' \geq y \vee y' \\ y \prec_{fuz} x \Rightarrow y \wedge y' \leq x \wedge x' \ \& \ y \vee y' \geq x \vee x' \end{array} \right\} \Rightarrow \left. \begin{array}{l} x \wedge x' = y \wedge y' \\ \& \\ x \vee x' = y \vee y' \end{array} \right\}$$

and this does not imply that, necessarily, $x = y$.

2.3 Proposition

If $L = [0, 1]$, it is proved in [6] that

$$x \prec_{fuz} y \iff x \leq^s y \text{ or } x \leq^s y'$$

where \leq^s is the sharpened order relation used in the Proposition 1.7.

Remark. It is immediate to prove that

1. The element 0.5 is the maximum element for this relation.

$$x \prec_{fuz} 0.5, \quad \forall x \in L$$

2. The elements 0 and 1 are the minimal elements.

$$\forall x \in L, \quad 0 \prec_{fuz} x, \quad 1 \prec_{fuz} x$$

3. The following equivalences are verified

$$x \prec_{fuz} y \iff x \prec_{fuz} y' \iff x' \prec_{fuz} y' \iff x' \prec_{fuz} y$$

In order to characterize the relation 'to be at least as fuzzy as' in the set of closed intervals in $L = [0, 1]$, we define on $\mathcal{J}[L]$ the following order relation.

2.4 Definition

Given two intervals $[a, b], [c, d] \in \mathcal{J}[L]$,

$$\begin{aligned} [a, b] \leq_{[0.5, 0.5]} [c, d] &\iff \\ &\iff \left\{ \begin{array}{l} [a, b] \wedge [0.5, 0.5] \leq [c, d] \wedge [0.5, 0.5] \\ \& \\ [a, b] \vee [0.5, 0.5] \geq [c, d] \vee [0.5, 0.5] \end{array} \right. \\ &\iff \left\{ \begin{array}{l} [a, b] \leq [c, d] \text{ if } [a, b], [c, d] \leq [0.5, 0.5] \\ a \leq c \ \& \ b \geq d \text{ if } a, c \leq 0.5 \ \& \ b, d \geq 0.5 \\ [a, b] \geq [c, d] \text{ if } [a, b], [c, d] \geq [0.5, 0.5] \end{array} \right. \end{aligned}$$

2.5 Proposition

The order relation $\leq_{[0.5, 0.5]}$ is an extension to the set of closed intervals in L of the 'sharpened order' \leq^s defined on L .

Proof The order relation $\leq_{[0.5, 0.5]}$ is the dual order of the ω_0 -activity order $\preceq_{[0.5, 0.5]}$ defined on set of intervals $\mathcal{J}[L]$ as

$$[a, b] \preceq_{[0.5, 0.5]} [c, d] \iff$$

$$\left\{ \begin{array}{l} [a, b] \wedge [0.5, 0.5] \geq [c, d] \wedge [0.5, 0.5] \\ \& \\ [a, b] \vee [0.5, 0.5] \leq [c, d] \vee [0.5, 0.5] \end{array} \right.$$

As we proved in the Proposition 1.8, the ω_0 -activity order between intervals $\preceq_{[0.5, 0.5]}$ is an extension of the order $\preceq_{0.5}$

$$([a, b] \preceq_{[0.5, 0.5]} [c, d]) \iff (a \preceq_{0.5} c) \ \& \ (b \preceq_{0.5} d)$$

Considering the dual orders and taking into account that $\preceq_{0.5}$ is the dual order of the 'sharpened order' \leq^s defined on $L = [0, 1]$, we obtain:

$$([a, b] \leq_{[0.5, 0.5]} [c, d]) \iff (a \leq^s c) \ \& \ (b \leq^s d)$$

That is, the order relation $\leq_{[0.5, 0.5]}$ is an extension to the set of closed intervals in L of the 'sharpened order' \leq^s . \square

2.6 Example

In Figure 3 we have represented the semilattice $(\mathcal{J}[\mathcal{C}], \leq_{[0.5, 0.5]})$ of the closed intervals in the chain $\mathcal{C} = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$.

The extension to the set of closed intervals in L of the order relation \leq^s allows us to extend the relation 'to be at least as fuzzy as' in the following way:

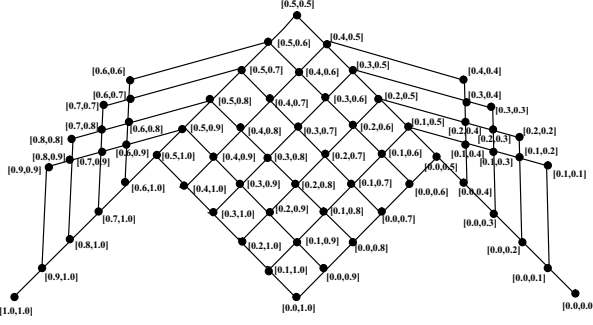


Figure 3: Semilattice $(\mathcal{J}[\mathcal{C}], \leq_{[0.5,0.5]})$

2.7 Definition

Given two intervals $[a, b], [c, d] \in \mathcal{J}[L]$, we will say that the interval $[c, d]$ is *at least as fuzzy as* the interval $[a, b]$, and we will denote it by:

$$[a, b] \prec_{fuz} [c, d] \iff \begin{cases} [a, b] \leq_{[0.5,0.5]} [c, d] \\ \text{or} \\ [a, b] \leq_{[0.5,0.5]} [1-d, 1-c] \end{cases}$$

that is:

$$[a, b] \prec_{fuz} [c, d] \iff \begin{cases} [a, b] \leq [c, d] \leq [0.5, 0.5] \\ \text{or} \\ a \leq c \leq 0.5 \ \& \ b \geq d \geq 0.5 \\ \text{or} \\ [a, b] \geq [c, d] \geq [0.5, 0.5] \\ \text{or} \\ [a, b] \leq [1-d, 1-c] \leq [0.5, 0.5] \\ \text{or} \\ a \leq 1-d \leq 0.5 \ \& \ b \geq 1-c \geq 0.5 \\ \text{or} \\ [a, b] \geq [1-d, 1-c] \geq [0.5, 0.5] \end{cases}$$

Remarks.

1. The interval $[0.5, 0.5]$ is at least as fuzzy as any other element of the set $\mathcal{J}[L]$.

$$[a, b] \prec_{fuz} [0.5, 0.5], \quad \forall [a, b] \in \mathcal{J}[L]$$

Therefore, it will be the interval that represents the greatest fuzziness.

2. The intervals $[0, 0]$, $[0, 1]$ and $[1, 1]$ are the minimal elements. However, it is not verified that any element is at least as fuzzy as the minimal elements, since they can be not comparable elements. For example, $[0, 0] \not\prec_{fuz} [0, 0.8]$ and $[0, 0.8] \not\prec_{fuz} [0, 0]$.

In next section we will try to do a similar study in order to be able to compare the ambiguity of the closed intervals in L .

3 Preorder relation associated to the ambiguity of the intervals

Intuitively, we associate the amplitude of the intervals to the ambiguity. The interval $[0, 1]$ is the one that has the biggest amplitude and so, following this idea, it will be the one that would represent the greatest ambiguity. For this reason, we will do a development similar to the one done in the previous section to compare the fuzziness between two intervals using an order relation for which the interval $[0, 1]$ is the maximum element.

3.1 Definition

Given two intervals $[a, b], [c, d] \in \mathcal{J}[L]$, the order relation $\leq_{[0,1]}$ is defined as:

$$[a, b] \leq_{[0,1]} [c, d] \iff \begin{cases} [a, b] \wedge [0, 1] \leq [c, d] \wedge [0, 1] \\ \& \\ [a, b] \vee [0, 1] \geq [c, d] \vee [0, 1] \end{cases} \iff \begin{cases} [0, b] \leq [0, d] \\ \& \\ [a, 1] \geq [c, 1] \end{cases} \iff \begin{cases} b \leq d \\ \& \\ a \geq c \end{cases}$$

The order relation $\leq_{[0,1]}$ agrees with the inclusion \subseteq between intervals:

$$[a, b] \leq_{[0,1]} [c, d] \iff [a, b] \subseteq [c, d]$$

Remark. The order relation $\leq_{[0,1]}$ is the order relation dual of the ω_0 -activity order $\preceq_{[0,1]}$, which, as we saw before, is the order of inclusion \supseteq between intervals.

3.2 Example

Let $(\mathcal{J}[\mathcal{C}], \leq)$ be the lattice of closed intervals in the chain $\mathcal{C} = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$, then the semilattice $(\mathcal{J}[\mathcal{C}], \leq_{[0,1]})$ can be represented by the graphic of Figure 4.

Using this order relation defined on the set of intervals, we will define the relation 'to be at least as ambiguous as'.

3.3 Definition

Given two intervals $[a, b], [c, d] \in \mathcal{J}[L]$, we will say that the interval $[c, d]$ is *at least as ambiguous as* the inter-

