

# Recent Advances in the Field of Left-continuous T-norms

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## Abstract

The recent advances in the research of left-continuous  $t$ -norms is summarized in this talk. The main focus is on construction methods, geometric description, and structural characterization. We point out further research directions and open problems.

## 1 Summary

The structure of continuous  $t$ -norms has been known since ages [19]. In many mathematical theories *left-continuous*  $t$ -norms have been used for a few decades without having a single (non-continuous) example at hand. After this long period – when no left-continuous  $t$ -norm<sup>1</sup> was known – the first example, the nilpotent minimum, was found by Fodor [2]. Even after finding this example many researchers believed that this is the only one, and a conjecture was published saying, roughly speaking, that the nilpotent minimum is the only left-continuous  $t$ -norm.

Next, a series of papers have appeared [11, 14, 15, 8, 7, 3, 1, 16, 5, 20, 6, 22], in which new construction methods provided the interested community with a huge number of new left-continuous  $t$ -norms. A state of the art is in [10].

On one hand, the complexity of different construction methods has reached a high level, therefore this line of research has become difficult to follow. On the other hand, the huge number of examples of left-continuous  $t$ -norms has called for a kind of systematization. One reply for that need is a comprehensive geometric characterization of residuated structures [9, 12], in particular, of left-continuous  $t$ -norms. This geometric description provides a kind of geometric view of algebraic phenomena, thus it makes the field of left-continuous  $t$ -norms, and its construction methods much more understandable. A recent result along this direction is [17], which resulted in another construction method in [18].

Concerning structural characterization, little is known as of today. As it will be explained, it seems to be hopeless to give classification for the whole class of left-continuous  $t$ -norms. However this does not exclude the possibility of characterizing certain subclasses of them. Two recent results concerns the cancellative, and the regular cases, respectively [4, 21].

These are the focuses of the talk.

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<sup>1</sup>From now under left-continuous  $t$ -norms we mean left-continuous but not continuous  $t$ -norms.

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