On a modeling of decision making with a twofold integral

Hideyuki Imai
Div. of Sys. and Info. Eng.
Hokkaido University
Sapporo, 060–8628 Japan
imai@main.eng.hokudai.ac.jp

Vicenç Torra
Institut d'Investigació
en Intel·ligència Artificial - CSIC,
Campus UAB s/n, 08193 Bellaterra
Catalonia, Spain.
vtorra@iiia.csic.es

Abstract

A Sugeno and a Choquet integrals are commonly used fuzzy integrals for aggregation. As a generalization of both integrals, the twofold integral is induced. The twofold integral enables us to interpret two measures from a different semantics viewpoint. One corresponds to the Choquet integral and the other corresponds to the Sugeno integral. Our work is about building models for the twofold integrals from examples.

In this work, we formulate the problem of learning measures from examples, and propose a method for obtaining the two fuzzy measures used in twofold integrals. This method is based on an alternate optimization.

Keywords: Twofold integral, Fuzzy measure, Model building, Optimization

1 Introduction

Modeling user decisions is an essential step in the process of building decision making applications. Aggregation operators are the basic tools for achieving such modeling. These operators, from a general perspective, combine data from different sources to build an aggregated value. In the case of decision making, they are usually applied to compute an overall assessment of a given alternative taking into account several (possible quite different) criteria.

Choquet integral and Sugeno integral are powerful aggregation operators that are specially appropriate when criteria are not independent but there exist some interaction between them. These operators combine the information taking into account a fuzzy measure. This measure permits to express the interaction (redundancy / complementariness) between the criteria.

Recently, Torra[12] proposed the twofold integral that generalizes both Choquet and Sugeno integrals. This new operator combines the evaluation of each criteria with respect to two fuzzy measures. One measure corresponds to the Choquet integral and the other corresponds to the Sugeno integral. The rationale of this generalization is that both measures have a different underlying semantics and thus a generalization using both is meaningful and useful. In fact, the measure in the Choquet is considered as more probability-like and the one in the Sugeno integral is considered more fuzzy-like.

Besides of the need of new aggregation operators, an important aspect for building decision making application is the process of constructing real models with these operators. One of the approaches considered for such modeling is parameter determination from examples. In this paper we work in this direction. We consider the process of parameter learning from examples for the twofold integral.

The structure of the paper is as follows. In Section 2, we describe a Choquet, a Sugeno, and a twofold integrals. Then in Section 3, we describe our approach for fuzzy measures determination. The paper finishes in Section 4 with some conclusions and future work.

2 A Choquet integral, a Sugeno integral and a twofold integral

We give the definition of three kind of integrals, a Choquet integral, a Sugeno integral, and a twofold integral. At first, we define a fuzzy measure because these integrals are defined by using a fuzzy measure.

Definition 1 *Let $X = \{x_1, \ldots, x_p\}$ be a finite set of criteria or attributes, and let $\mathcal{F}$ be the family of all subset of X. A fuzzy measure μ on the finite set X is a set function $\mu : \mathcal{F} \to [0,1]$ satisfying the following conditions:*

1. $\mu(\emptyset) = 0, \mu(X) = 1$ *(boundary condition)*
2. $A \subset B$ *implies* $\mu(A) \leq \mu(B)$ *for all* $A, B \in \mathcal{F}$ *(monotonicity)*

To determine a fuzzy measure on X, we must identify $2^p - 2$ coefficients satisfying $p2^{p-1}$ conditions. In practical situations, these determination is not easy. To solve this drawback, some approach have been proposed to reduce the number of parameters to be determined[6, 10]. In this paper, we use a Sugeno λ-measure defined as follows:

Definition 2 *Let μ be a fuzzy measure, then μ is a Sugeno λ-measure if there exists $\lambda > -1$ such that*

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda \mu(A) \mu(B) \qquad (1)$$

holds for all $A, B \in \mathcal{F}$.

It is to be noted that, for a Sugeno λ-measure,

$$\prod_{j=1}^{p} (1 + \lambda \mu(\{x_j\})) = 1 + \lambda \qquad (2)$$

holds because of the boundary condition. In [11], it is shown that a Sugeno λ-measure is completely determined by the measures of singleton, $\mu(\{x_i\}), i = 1, \ldots, p$, as

$$\mu(A) = \frac{1}{\lambda} \{ \prod_{x_j \in A} (1 + \lambda \mu(\{x_j\})) - 1 \} \qquad (3)$$

for all $A \in \mathcal{F}$.

The Choquet and the Sugeno integrals integrates a function f with respect to a fuzzy measure.

Definition 3 *Let μ be a fuzzy measure on X, then a Choquet integral $CI_\mu(f)$ and a Sugeno integral $SI_\mu(f)$ of a function $f : X \to [0,1]$ with respect to the fuzzy measure μ are defined by*

$$CI_\mu(f) = \sum_{j=1}^{p} \{ f(x_{s(j)}) - f(x_{s(j-1)}) \} \mu(A_{s(j)})$$

and

$$SI_\mu(f) = \bigvee_{j=1}^{p} \{ f(x_{s(j)}) \wedge \mu(A_{s(j)}) \}$$

respectively, where $f(x_{s(i)})$ indicates that the indices have been permuted so that $0 \leq f(x_{s(1)}) \leq \cdots \leq f(x_{s(p)}) \leq 1$, $A_{s(j)} = \{x_{s(j)}, \ldots, x_{s(p)}\}$ and $f(x_{s(0)}) = 0$.

For detail of these two integrals, see[1, 3, 5, 10].

As a generalization of the Choquet and the Sugeno integrals, Murofushi and Sugeno[8] defined a t-conorm integral. It is built through the unification of the operations addition and maximum in terms of t-conorms, and product and minimum in terms of a product-like operation. Additionally, it is important to underline that the fuzzy measure present in both integrals is considered the same measure and, therefore, the t-conorm integral considers a single fuzzy measure.

A twofold integral, defined in [12], is a completely different generalization, because the cornerstone of a construction is to consider fuzzy measures in Sugeno and Choquet integrals as completely different from a semantics point of view. In particular, we consider Sugeno's measure as denoting fuzziness and Choquet's measure as denoting randomness.

From this assumption, it is natural to infer that a twofold integral has to be an expression that contains the two measures. Then, naturally, particularizations of the measures should lead either to the Choquet integral or the Sugeno integral. When nothing is known about the membership of X but only information about randomness is known, then the integral to be used is Choquet's one. Instead, when nothing is known about the probability distribution but only the fuzziness then the integral to be used is Sugeno's one.

The following is the definition of a twofold integral. Let μ_C be a fuzzy measure in the Choquet integral and let μ_S be one in the Sugeno integral, then the twofold integral which is denoted by $TI_{\mu_S,\mu_C}(f)$, is defined as follows:

Definition 4 *Let μ_C and μ_S be two fuzzy measures on X, then the twofold integral of a function $f : X \to [0,1]$ with respect to the fuzzy measures μ_S and μ_C is defined by:*

$$\begin{aligned} TI_{\mu_S,\mu_C}(f) &= \sum_{j=1}^{p} [\{ \bigvee_{k=1}^{j} f(x_{s(k)}) \wedge \mu_S(A_{s(k)}) \} \\ &\quad \times \{ \mu_C(A_{s(j)}) - \mu_C(A_{s(j+1)}) \}] \end{aligned}$$

where $f(x_{s(j)})$ indicates that the indices have been permuted so that $0 \leq f(x_{s(1)}) \leq \cdots \leq f(x_{s(p)}) \leq 1$, $A_{s(j)} = \{x_{s(j)}, \cdots, x_{s(p)}\}$, $A_{s(p+1)} = \emptyset$.

For detail of a twofold integral, see [12].

3 Fuzzy measures identification problem with a twofold integral

We propose a method for obtaining two fuzzy measures from example. Let $y_1, \ldots, y_N$ be global evaluations of N objects (or by N individuals), and let $f_1(x_j), \ldots, f_N(x_j), j = 1, \ldots, M$, be their evaluations of criterion x_j. Our aim is to determine measures μ_C and μ_S on $X = \{x_1, \ldots, x_p\}$ such that they minimize

$$g(\mu_S, \mu_C) = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2, \qquad (4)$$

where

$$\hat{y}_i = TI_{\mu_S, \mu_C}(f_i).$$

In this study, we use a Sugeno λ-measure as μ_S. Thus, the number of coefficients to be determined is $2^p + p - 2$, and the model $\hat{y}_i$ becomes

$$\hat{y}_i = \sum_{j=1}^{p} [\{ \bigvee_{k=1}^{j} f_i(x_{i,s(k)}) \wedge \mu_S(A_{i,s(k)}) \} \times \{\mu_C(A_{i,s(j)}) - \mu_C(A_{i,s(j+1)})\}] \qquad (5)$$

where $f_i(x_{s(j)})$ indicates that the indices have been permuted so that $0 \leq f_i(x_{i,s(1)}) \leq \cdots \leq f_i(x_{i,s(p)}) \leq 1$, $A_{i,s(j)} = \{x_{i,s(j)}, \cdots, x_{i,s(p)}\}$, $A_{i,s(p+1)} = \emptyset$, and μ_S is a Sugeno λ-measure.

To obtain the optimum measures μ_S and μ_C, we propose an alternate optimization method because it is not easy to simultaneously obtain both measures. The proposed algorithm is as follows:

Step 0 : (Initialization) Set $\mu_S^{(0)}(\{x_j\})$ for all $x_j \in X$ such that $0 < \mu_S^{(0)}(\{x_j\}) < 1$ as an initial values of a Sugeno λ-measure, and set $k := 0$. Also set $\delta > 0$ which is used for evaluation of convergence of the algorithm.

Step 1 : (Calculation of the Sugeno λ-measure) Obtain λ satisfying the equation (2), and calculate $\mu(A)$ for all $A \in X$ according to the equation (1). The obtained measure is denoted by $\mu_S^{(k+1)}$.

Step 2 : (Determination of μ_C) Determine the measure μ_C under $\mu_S^{(k+1)}$ being fixed, and the obtained measure is denoted by $\mu_C^{(k+1)}$.

Step 3 : (Determination of $\mu_S(\{x_j\})$) Determine measures of singleton $\mu_S(\{x_j\}), j = 1, \ldots, p$ under $\mu_S^{(k+1)}$ being fixed, and the obtained measures is denoted by $\mu_S^{(k+1)}(\{x_j\}), j = 1, \ldots, p$. Also calculate

$$\hat{y}_i^{(k+1)} = TI_{\mu_S^{(k+1)}, \mu_C^{(k+1)}}(f_i).$$

Step 4 : (Evaluation of convergence) If

$$||\hat{y}_i^{(k+1)} - \hat{y}_i^{(k)}|| < \delta$$

then end the algorithm, else set $k := k + 1$ and goto *Step 1*.

Step 2 and Step 3 are executed as follows.

In Step 2, once μ_S is fixed, the equation (5) becomes

$$\hat{y}_i = \sum_{j=1}^{p} [\{ \bigvee_{k=1}^{j} f(x_{i,s(k)}) \wedge \mu_S(A_{i,s(k)}) - \bigvee_{k=1}^{j-1} f(x_{i,s(k)}) \wedge \mu_S(A_{i,s(k)}) \} \mu_C(A_{i,s(j)})].$$

This is the problem of a fuzzy measure identification with the Choquet integral, and we can use the method of the algorithm for the problem [4, 6, 9]. To construct algorithms, we need to know the properties of the solution set, for example, shown in [6, 7].

Since this optimization problem is a convex quadratic programming problem, the complementary pivot algorithm can be applied[9]. In [6], a method based on alternate convex projections[2] is proposed. In our implementation, we use the method based on alternate convex projections.

In Step 3, we can not use a usual gradient method to obtain the optimum measure μ_S, because the objective function is not differentiable at finite points.

Since μ_C is fixed, the objective function (4) is a function of μ_S. Moreover, since we use a Sugeno λ-measure as μ_S, the objective function is a function of $v_1, \ldots, v_p$. Thus, we rewrite the objective function in this step as follows.

Let $v = [v_1, \ldots, v_p]$ where $v_j = \mu_S(x_j)$, then $\mu_S(A), A \in \mathcal{F}$ is the function of v by the equation (3), which is denoted by $Q(v)$, and the objective function (4) with fixed μ_C, which is regarded as the function of v, is denoted by

$$g(v|\mu_C) = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2,$$

Thus the sub-algorithm is as following :

Step 3.0 : Set $\varepsilon > 0$, which is sufficiently small, and let $\mu_C(A) := \mu_C^{(k)}(A), A \in X$, $v_j := \mu_S^{(k)}(x_j), j = 1, \ldots, p$.

Step 3.1 : Calculate

$$\Delta_\alpha^+ = g(v_1, \ldots, v_\alpha + \varepsilon, \ldots, v_p|\mu_C) - g(v|\mu_C),$$
$$\Delta_\alpha^- = g(v_1, \ldots, v_\alpha - \varepsilon, \ldots, v_p|\mu_C) - g(v|\mu_C)$$

and set $\Delta_\alpha = \min(\Delta_\alpha^+, \Delta_\alpha^-, 0)$.

Step 3.2 : Solve the one dimensional optimization problem, which is to minimize the function

$$g(v + t[\Delta_1, \ldots, \Delta_p]|\mu_C).$$

Let

$$t^* = \arg\min_{t>0} g(v + t[\Delta_1, \ldots, \Delta_p]|\mu_C).$$

and $v_\alpha^* = v_\alpha + t^*[\Delta_1, \ldots, \Delta_p]$

Step 3.3 : If $t^* < \delta$, then set $\mu_S^{(k+1)}(x_j) := v_j^*, j = 1, \ldots, p$ and exit the sub-algorithm, else set $v_j := v_j^*, j = 1, \ldots, p$ and goto *Step 3. 1*.

4 Conclusions and future work

A twofold integral is a generalization of both a Sugeno and a Choquet integrals. In this paper, we formulated the problem of identification of fuzzy measures in decision making with a twofold integral. In addition, we proposed a optimization method, which is a kind of alternate optimization technique, to identify these fuzzy measures.

The proposed method has been practically applied to the iris data in the UCI repository. These experiments are not reported here because of the lack of space. Preliminary results show that the method is promising although additional experimentation and a more detailed analysis are required.

Acknowledgements

This work is partly supported by the grant of the Ministry of Education, Culture, Sports, Science and Technology, Japan.

References

[1] G. Choquet, (1953). Theory of Capacities. *Ann. Inst. Fourier*, 5:131–296.

[2] J. P. Boyle and R. Dykstra, (1985). A method of finding projections onto the intersection of convex sets in Hilbert spaces. *Advances in Order Restricted Statistical Inference*, pages 28–47. Springer-Verlag.

[3] D. Denneberg, (1994). *Non-additive Measure and Integral.* Kluwer.

[4] M. Grabisch, (1995). A new algorithm for identifying fuzzy measures and its application to pattern recognition. In *Proceedings of 4th FUZZ-IEEE*, pages 145–151, Yokohama.

[5] M. Grabisch, H. T. Nguen and E. A. Walker (1995). *Fundamentals of Uncertainty Calculi with Application to Fuzzy Inference.* Kluwer.

[6] H. Imai, M. Miyamori, M. Miyakoshi, and Y. Sato, (2000). An algorithm based on alternate projections for a fuzzy measure identification problem. *Proc. Int. Conf. Soft Comp.*, Iizuka.

[7] P. Miranda and M. Grabisch, (1999). Optimization issues for fuzzy measures. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 7(6) : 545–560.

[8] T. Murofushi and M. Sugeno, (1991). Fuzzy t-conorm integral with respect to fuzzy measures: Generalization of Sugeno integral and Choquet integral. *Fuzzy Sets and Systems*, 42(1) : 57–71.

[9] Y. Nakamori, (1998). Identification and interpretation of fuzzy measures. *Journal of Japan Society for Fuzzy Theory and Systems*, 10(2) : 215–224. in Japanese.

[10] M. Sugeno, (1974). Theory of fuzzy integrals and its application. Ph.D Dissertation, Tokyo Institute of Technology, Tokyo, Japan.

[11] H. Tahashi and J. M. Keller, (1990), Information fusion in computer vision using the fuzzy integral. *IEEE. Trans. on Sys. Man. & Cyber.*, 20(3) : 733–741.

[12] V. Torra. Twofold integral : A Choquet integral and Sugeno integral generalization. submitted.