Nearness and Uniform Convergence

Vladimír Janiš
Dept. of Mathematics, Fac. of Science
Matej Bel University, Banská Bystrica
Slovak Republic
janis@fpv.umb.sk

Abstract

Nearness (a fuzzy nearness) is a fuzzy relation that can be used to model various grades of "being close" in a linear space. We study the uniform convergence of a sequence of functions with values in a space equipped with a nearness relation. The uniform convergence for the mappings into a space with a fuzzy nearness is defined and it is shown that a theorem similar to Moore-Osgood theorem for crisp case holds.

Keywords: Fuzzy nearness, uniform convergence.

1 Fuzzy equivalences

Let X be a nonempty set, let T be a triangular norm. A fuzzy relation $E : X^2 \to [0,1]$ is called a T-equivalence if

- $E(x,x) = 1$ for all $x \in X$,
- $E(x,y) = E(y,x)$ for all $x, y \in X$,
- $T(E(x,y), E(y,z)) \leq E(x,z)$ for all $x, y, z \in X$.

The third property corresponds to transitivity of a (crisp) equivalence relation.

In [1] the authors claim that namely this property is a reason why T-equivalences are not suitable to model some real-life situations of approximate equality. An easy example is a consideration to what extent two persons are of about the same age. If $E(x,y)$ denotes the grade of acceptance for the statements "persons aged x and y are of about the same age" then the inequality

$$T(E(20,25), E(25,30)) \leq E(20,30)$$

is obviously not what we expect, especially if we use the minimum triangular norm.

2 The relation of fuzzy nearness

In fact the problem mentioned in the previous example is a litte artificial, as it mixes together two different points of view - that of a fuzzy logic and that of a metric. Nevertheless, as an alternative to T-equivalences the authors in [1] define so called resemblance relation R in a metric space (X, ρ) which has the first and the second property of a fuzzy equivalence (hence it is a fuzzy compatibility) while the third one is replaced by the condition

$$R(x,z) \leq R(x,y) \text{ whenever } \rho(x,z) \geq \rho(x,y).$$

A fuzzy relation with the same properties called a fuzzy nearness has been introduced by Kalina in [4] and studied in [2, 3, 7]. The definition of a fuzzy nearness in a linear space is the following (see [8]):

Definition 1 *Let X be a linear space. A mapping $N : X \times X \to [0,1]$ is a relation of fuzzy nearness, if the following hold:*

1. *$N(x,x) = 1$ for any $x \in X$,*
2. *$N(x,y) = N(y,x)$ for any $x, y \in X$,*

3. *for all $x, y \in X$ and any real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$*

$$N(x + t_1 y, x + t_4 y) \leq N(x + t_2 y, x + t_3 y),$$

4. *for all $x, y \in X$ the following holds*

$$\lim_{t \to \infty} N(x, x + ty) = 0.$$

If no confusion can occur, we use the term nearness instead of a fuzzy nearness. We restrict ourselves to pseudometric spaces, hence all the mappings we use are mappings between pseudometric spaces. Finally, as we work in pseudometric space with a pseudometric d, by a nearness we understand a reflexive and symmetric mapping $N : X^2 \to [0, 1]$ such that $N(x, y) \leq N(x, z)$ whenever $d(x, z) \leq d(x, y)$. This is a special case of the definition above.

3 α-limits and uniform convergence

In [5, 6, 7] the notion of a fuzzy limit has been studied. It is based on the following notion of an α-limit:

Definition 2 *Let $f : X \to Y$, let N be a fuzzy nearness on Y, let $\alpha \in]0, 1]$. Then y is an α-limit of f at $x_0 \in X$ with respect to N if there is a δ-neighborhood of x_0 such that for each x from this δ-neighborhood, $x \neq x_0$ the inequality*

$$N(f(x), y) \geq \alpha$$

holds.

In fact the author in [5] uses a more general approach considering also a fuzzy nearness in X, but for our purposes we restrict ourselves only to the case of a nearness in the space Y.

Analogically it is possible to define the α-limit of a sequence, see also [5].

If the α-limit at a point exists, it need not be unique, as it is shown in the following example.

Example. Let N be the following nearness on R (the set of all reals): for $x, y \in R$ put $N(x, y) = \max\{1 - |x - y|, 0\}$. Let $f(x) = 0$ for all $x \in R$, let $x_0 \in R, 0 < \alpha \leq 1$. Then any number from the interval $[\alpha - 1, 1 - \alpha]$ is an α-limit of f at x_0.

This weakened concept of a limit has as a natural consequence (due to Heine-Borel theorem) a weaker form of the continuity.

Definition 3 *Let $f : X \to Y, x_0 \in X, 0 < \alpha \leq 1$. Then f is α-continuous at x_0 if $f(x_0)$ is an α-limit of f at x_0. We say that f is α-continuous on a set, if it is α-continuous at each its point.*

One of the crucial points in the study of mappings between (pseudo)metric spaces is the uniform convergence. Its importance is mainly in the possibility of "limits replacement" which follows from the following proposition, (rarely) known as the Moore-Osgood theorem.

Proposition 1 *Let X, Y be pseudometric spaces, let $f, f_n : X \to Y$ for all natural n, let $x_0 \in X$. If the sequence f_n uniformly converges to f on X and $\lim_{x \to x_0} f_n(x) = a_n$ for all natural n , then $\lim_{n \to \infty} a_n = f(x_0)$.*

Although it is possible to define uniform convergence for the spaces with a nearness in various ways, in order to guarantee the validity of the above proposition we choose the following:

Definition 4 *Let X, Y be pseudometric spaces, let $f, f_n : X \to Y$ for all natural n, let N be a nearness on Y. We say that the sequence (f_n) uniformly converges to f on X with respect to N if for all $\alpha \in]0, 1[$ there is a natural n_0 such that for all $n \geq n_0$ and all $x \in X$ the inequality*

$$N(f_n(x), f(x)) > \alpha$$

holds.

With this definition of the uniform convergence we obtain the following version of the Moore-Osgood theorem:

Proposition 2 *Let X, Y be pseudometric spaces, N be a nearness in Y, let $f, f_n : X \to Y$ for all natural n, let $x_0 \in X$. If the sequence f_n uniformly converges to f on X with respect to N, and a_n is an α-limit of f_n at x_0, then $f(x_0)$ is an α-limit of the sequence (a_n).*

The proof of the proposition is based on the properties of α-limits of functions and sequences. This proposition enables us to formulate conclusions on the limit function of a uniformly convergent series similar to those valid in the classical case, like e.g. continuity of the limit function, possibility of termwise differentiation and integration, etc.

Acknowledgements

The work was supported by grant 1/9056/02 of Slovak Grant Agency VEGA. The author would like to express his gratitude to the Department of Statistics and Operation Research, University of Oviedo, Spain, for excellent conditions for this research.

References

[1] DeCock, M., Kerre, E.: *On (un)suitable fuzzy relations to model approximate equality*, Fuzzy Sets and Systems.

[2] Dobrakovová, J.: *On a fuzzy nearness*, In: Proceedings of the conference Strojné inžinierstvo 1, 1998, 33-37.

[3] Dobrakovová, J.: *Nearness, convergence and topology*, BUSEFAL **80** (1999), 17-23.

[4] Kalina, M.: *Derivatives of fuzzy functions and fuzzy derivatives*, Tatra Mountains Math. Publ. **12** (1997), 27-34.

[5] Kalina, M.: *Fuzzy limits and fuzzy nearness relation*, In: Proc. of 7th Fuzzy Days in Dortmund, October 2001, Springer, LNCS 2206, 755-761.

[6] Kalina, M.: *Nearness-based limits in multidimensional case*, Abstracts of 23rd Linz Seminar on Fuzzy Sets Theory, Analytical Methods and Fuzzy Sets, 19.

[7] Kalina, M., Dobrakovová, J.: *Relation of fuzzy nearness in Banach space*, Proc. East-West Fuzzy Colloquium, IPM Zittau, 2002, 26-32.

[8] Kalina, M.: *Nearness Relation and Topology*, this volume.