The role and properties of rule relevancy in fuzzy inference process

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Abstract

Fuzzy inference process usually involves the use of fuzzy rule base consisting in several fuzzy rules. Overall output can be obtained by aggregation of outputs of all rules. To obtain an output of individual rule the relevancy of this rule is calculated. Then the individual output is obtained from the relevancy and the consequent of the rule. Such process can be realised using an operator that is called the relevancy transformation operator. It must fulfil some requirements, which are tightly connected with an aggregation step. The properties of such operators are studied.

Keywords: Implication, Conjunction, Relevancy transformation operator, Information Boundedness principle, Measure of information.

1 Introduction

The fuzzy expert systems usually involves fuzzy inference engine operating on a series of fuzzy rules, most commonly having the form “If A then B”, where A and B are fuzzy sets. Overall output can be obtained by aggregation of outputs of all individual rules and by defuzification. To obtain an output of individual rule the relevancy of this rule can be calculated from given input data and the antecedent of the rule. It allows to avoid conflicting rules. Between rules that conflict with one another, it prefers the one with the largest relevancy. The individual output is obtained from the relevancy and the consequent of the rule. Such process can be realized using the operator, which is sometimes called the relevancy transformation operator. It must fulfill some requirements, which are tightly connected with an aggregation step. E.g., if the rule relevancy is zero, the effective output should not influence the aggregated result. From another point a view, it can be considered as a generalization of fuzzy implications and fuzzy conjunctions.

2 Preliminaries

A basic operation usually performed in approximate reasoning is the operation of fuzzy modus ponens. Let U and V be two variables taking over the spaces X and Y, respectively. Let A and A’ be fuzzy sets over spaces X and B and B’ be fuzzy sets over the space Y. The simple pattern of fuzzy modus ponens has form:

Propositions: If U is A then V is B

U is A’

Conclusion V is B’

The fuzzy sets A and B usually correspond to some linguistic concepts. They are usually called the antecedent and the consequent of the rule, respectively. The fuzzy sets A’ and B’ are usually called the input and the output of the rule. The process of obtaining the input B’ of the rule consist in:

1. Find a matching value r of given rule from the input A’ and the antecedent A
2. Find an inferred fuzzy set B’ from matching value r and the antecedent B

There exist many way how to obtain the matching value r. E.g., if the input is a crisp value x^* then the matching value can be the membership value of x^*

$$r = A(x^*)$$

In the case of fuzzy input A’ the matching value can be obtain by

$$r = Sup[Min(A(x), A'(x))]$$

Then inferred fuzzy set are usually obtained pointwisely by

$$B'(y) = h(r, B(y))$$

The operator h is a generalization of classical implication or conjunction. In classical logic, these two operations coincide if the antecedent is fulfilled.

The fuzzy system model usually contains a collection of n rules of the form

$$\text{If U is } A_i \text{ then V is } B_i$$

The overall fuzzy output is obtained by aggregation of individual outputs. The main object of our consideration will be the operator h, which could be the generalization of fuzzy implication and fuzzy conjunction [2], respects the requirements of aggregation and has additional reasonable properties.

3 Relevancy transformation operators

Definition 1 Let $c \in [0; 1]$ be a given element. A binary operation $h: [0; 1]^2 \to [0; 1]$ is called a relevancy transformation (RET) operator with respect to the element c if it satisfies the following axioms, for all $a, a_1, a_2, r, r_1, r_2 \in [0; 1]$:

(1) $h(1, a) = a$

(2) $h(0, a) = c$

(3) $h(r, a_1) \le h(r, a_2)$ for all $a_1 < a_2$

(4) if $a \ge c$, then $h(r_1, a) \le h(r_2, a)$ for all $r_1 < r_2$

(5) if $a \le c$, then $h(r_1, a) \ge h(r_2, a)$ for all $r_1 < r_2$

The RET operator was introduced by R. Yager [6], [7], [8]. For more details see also [3]. The interesting example of a RET operator with respect to given $c \in [0, 1]$ is the product RET operator (PRET):

$$h(r, a) = r.a + (1-r) . c; \quad a,b \in [0, 1]$$

The axiom (2) says that if the rule relevancy is zero the individual output should not bring any information and should not influence the following aggregation step. The element c should be the neutral element under the aggregation operation. The next requirement on RET operator is that the knowledge obtained as a result of inference process should not have more information than that contained in the consequent of the rule, i.e.,

$$Inf(B') \le Inf(B)$$

where Inf is some measure of information. It is called Information Boundedness Principle (IBP). In the case of finite spaces X and Y, the measure of information can be obtained from a possibility distribution. It suggests that the maximal information is brought by singleton. It means that we know exactly the value of the variable. On the other hand the information is minimal if all elements have the same possibilities. There exist many ways how to define measure of information. We shall use the specificity measure [1], [5], [6], [7], [8] as a measure of information.

Consider fuzzy subsets of a finite universe X with the cardinality n. Then each fuzzy set can be represented by n-tuple

$$(a_1, a_2, \dots, a_n)$$

of membership values. Now we shall define the specificity measure as a measure of information on the system of all n-tuples $(a_1, a_2, \dots, a_n) \in [0; 1]^n$.

Definition 2 . A mapping $Sp : [0; 1]^n \to [0; 1]$ is called a specificity measure (specificity for short) if it satisfies the following axioms:

(1) For any permutation $(p1, p2, \dots, pn)$ of $(1, \dots, n)$ is
$Sp(a_1, a_2, \dots, a_n) = Sp(a_{p1}, a_{p2}, \dots, a_{pn})$

(2) If $1 \ge b_1 > b_2 \ge a_2 \ge a_3 \dots \ge a_n \ge 0$ then
$Sp(b_1, a_2, \dots, a_n) > Sp(b_2, a_2, \dots, a_n)$

(3) If $1 \ge a_1 \ge a_2 \ge a_3 \dots \ge a_n \ge 0$, $a_1 \ge b_2 \ge b_3 \ge \dots \ge b_n \ge 0$ and $a_i \ge b_i$ for each $i = 2, 3, \dots, n$, then
$Sp(a_1, a_2, \dots, a_n) \le Sp(a_1, b_2, \dots b_n)$

(4) $Sp(a_1, a_2, \dots, a_n) = 1$ if and only if there exists the unique i such that $a_i = 1$ and $a_j = 0$ for $j \ne i$

(5) $Sp(0, 0, \dots, 0) = 0$

Moreover the specificity is <u>shift invariant</u> if
$Sp(a_1, a_2, \dots, a_n) = Sp(a_1 + b, a_2 + b, \dots, a_n + b)$
for all $b \in [-\min(a_i), 1 - \max(a_i)]$.
The specificity

$$Sp(a_1, a_2, \dots, a_n) = a_1 - \sum_{i=2}^{n} \omega_i a_i$$

where $\omega_2 + \omega_3 + ... + \omega_n = 1$ is called linear specificity. Note that every linear specificity is shift invariant and Sp(a, a, …, a) = 0 for all $a \in [0, 1]$. For simplicity we shall use the abbreviation

$$h(r, B) = (h(r, b_1), h(r, b_2), \dots, h(r. b_n))$$

Definition 3 We shall say that the RET operator h satisfies IBP with respect to a given specificity Sp if for all relevancies $r_1, r_2 \in [0; 1]$, $r_1 < r_2$ and all $B \in [0; 1]^n$

$$Sp(h(r_1, B)) \leq Sp(h(r_2, B))$$

The following theorems give some conditions for IBP in the case n = 2 and can be generalized for all finite n.

Theorem 1. Let h be a continuous RET operator with respect to given element c and the specificity measure Sp be continuous on the unit square such that Sp(b, b) = 0 for all $b \in [0, 1]$. If for any $(x_0, y_0) \in [0, 1]^2$, $x_0 \neq y_0$ and for any $k \in (0, 1)$ has the equation

$$Sp(h(r, x_0), h(r, y_0)) = k$$

the unique solution, then h satisfies IBP with respect to the specificity Sp.

Theorem 2. A RET operator h satisfies IBP with respect to a shift invariant specificity Sp if and only if h is 2-increasing.

Recall that h is two increasing if

$$h(r_1 , a_1) + h(r_2 , a_2) \geq h(r_1 , a_2) + h(r_2 , a_1)$$

for all $0 \leq r_1 < r_2 \leq 1$, $0 \leq a_1 < a_2 \leq 1$. Theorem 2 implies that PRET satisfies IBP with respect to any shift invariant specificity. A two-dimensional copula [3] is a mapping $C : [0; 1]^2 \rightarrow [0; 1]$, that is two increasing and C(r, 0) = C(0, b) for all $r, b \in [0; 1]$. It means, that the copula is a RET operator with respect to the element c = 0. Theorem 2 also implies that the copula is a RET operator, which satisfies IBP with respect to any shift invariant specificity. Moreover the RET operator generated by a copula C by

$$h(r, a) = C(r, a) + C(1-r, c), \quad a, r \in [0, 1]$$

is a RET operator with respect to given $c \in [0, 1]$ It is two increasing and thus it fulfils IBP with respect to any shift invariant specificity.

The IBP property has simple geometric interpretation. Let $A = (a_1, a_2, \dots, a_n)$ be a fuzzy set with finite support. Let h be a RET operator. Consider set

$$Trj(h, A) = \{h(r, A), r \in [0, 1]\}$$

as a trajectory of some moving object. The starting point A corresponds to the relevancy r =1 and the terminal point B = (c, c,…, c) corresponds to the relevancy r = 0. Then for continuous RET operator h Trj(h, A) is a curve contained in n dimensional cube. The IBP principle means that the specificity is non decreasing on Trj(h, A) in the sense that for r < s : Sp(h(r, A)) ≤ Sp(h(s,A)).

The characterization of RET operators fulfilling IBP with respect to given specificity is a serious problem. The main problem is that for given trajectory Trj(h, A) and given point B from Trj(h, A), the trajectory Trj(h, B) is not necessarily contained in the trajectory Trj(h, A). It means that some trajectories can branch. To simplify this problem we shall define RET operators trajectories of which don't branch.

Definition 4 The operator $h: [0 ; 1]^2 \rightarrow [0 ; 1]$ is called in-line RET operator if for any $A \in [0; 1]^n$ and for any $B \in Trj(h, A)$ holds

$$Trj(h, B) \subseteq Trj(h, A)$$

The following theorem gives some possibility to construct an in-line RET operator

Theorem 3 Let $f : [0; 1] \rightarrow [-1; 1]$ be continuous, strictly increasing and c be an arbitrary element of the interval [0; 1] such that f(c) = 0. Then the operator

$$h(r, a) = f^{-1}(r.f(a)), \quad (r, a) \in [0; 1]^2$$

is in-line RET operator with respect to the element c.

The full characterization of in-line RET operator is still an open problem.

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