Analysis of Multi-product Break-even with Uncertain Information*

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Abstract

We revise the classic methodology to find the multi-product break-even point. In the current paper we propose a solution to the problem under uncertainty conditions, based on Durán Herrera's crisp approach [3] and applying fuzzy linear programming.
We introduce the concept of *approximate break-even* for cases in which, due to the nature of the problem, it is not possible to find the break-even point.
We propose two situations to deal with *approximate break-even* when the information regarding marginal contributions and /or total fixed costs is uncertain. A fuzzy linear programming method is applied to the study of a real case in a small enterprise. The characteristics of this case make it necessary to use integer linear programming.

Keywords: break-even point, multiproducing firm, uncertainty, fuzzy linear programming.

1. Introduction

After the production process begins in a firm, as long as sale prices are higher than variable costs, sale incomes keep absorbing part of the fixed costs up to the point where certain volumes of production and sales cover the fixed and variable costs. The point where sales equal expenses is called break-even point. There is no profit made or loss incurred at the break- even point.
The calculation of this point provides a useful tool to estimate profit based on sales volume, in the short term, since it allows to easily estimate expenses for any business operation level.
The uncertainty about fixed and variable production costs is a key aspect of administrative decision making. By calculating the break-even point we can analyze the importance of each of these variables and the way they affect each other.
The majority of the existing papers on the subject deal exclusively with companies producing and selling a single product or several products with a single marginal contribution. In practice these cases are a minority compared to firms with a wider variety of products and/or services. The analysis of the break-even point is not usually easy in these situations.
We revise the classic methodology to find the multi-product break-even point. In the current paper we propose a solution to the problem under uncertainty conditions, based on Durán Herrera's approach [4] applying fuzzy linear programming.
We introduce the concept of approximate break-even for cases in which, due to the nature of the problem, it is not possible to find the break-even point.
By using a general model for fuzzy linear programming developed by Delgado, M., Verdegay, J.L and Vila, M.A. [2] we study *approximate break-even* when the information regarding marginal contributions and/or total fixed costs is uncertain. The proposed method is applied to the study of a real case in a small enterprise. The characteristics of this case make it necessary to use integer linear programming. The information was obtained by interviewing the owners of the company. Data was processed using a specialized software application for integer linear programming.
Let us now introduce the notation needed in the rest of the paper. We will place a symbol ~ over a capital letter if it represents a fuzzy number so $\tilde{F}, \tilde{M}, \tilde{W}$ are all fuzzy numbers f, p, q, m will denote real numbers.

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2. Multi-product Break-even under Certainty Conditions

Cases of monoproducing or monoservicing firms are not as frequent as those of firms with a wider range of products and/or services.
In fact, what we are looking for is that the marginal contribution obtained by selling all the products - whichever their combination is - turned out to be equal to the total fixed costs.
Break-even analysis is generally not simple when it must be done in multiproducing firms, especially when there are technical conditionings, fund shortage or other type of restrictions.
It is possible to find different combinations of the products satisfying the break-even condition.
Many authors such as Yardin [11], Drimer [3] and Durán Herrera [4] have studied crisp multi-product break-even.
The current proposal is based on Durán Herrera's general approach [4] to the problem of calculating the break-even point in a multiproducing firm.
The following crisp linear programming problem models this situation.

Minimize: $I = \sum_{i=1}^{n} p_i \cdot q_i$

Subject to: $\sum_{i=1}^{n} (p_i - w_i).q_i = f$ (1)

$A.q_i : : B$ (2)

$q_i \geq 0$ (3)

where:
p_i: unit price for product i
q_i : amount of product i
I: sale income from the set of products i in currency units
f: total fixed costs
w_i: variable unit cost for product i
Analyzing the meaning of the restrictions we find:
(1) general break-even condition for the set of products i the firm produces.
(2) restrictive technical, commercial or financial conditions, if there were any.
(3) non-negativity of the decision variables

We look for the combination of the products satisfying the break-even condition that minimizes the sale income and fulfills the constraints.

3. Multi-product Break-even under Uncertainty Conditions

We study situations where the information is not known with accuracy and it can be expressed by fuzzy numbers.
As total fixed costs or unit variable costs could be imprecise, we use the general model for fuzzy linear programming proposed by Delgado, M., Verdegay, J.L. and Vila, M.A. [2].
This model of fuzzy linear programming allows to deal with the approaches which involve fuzzy constraints and those which involve fuzzy coefficients and gives a resolution method for them and all particular problems that may be deduced from it.
The problem to consider is:

Maximize $c.x$
Subject to $A.x \lesssim b$ (4)
$x \geq 0$

Where A is an $m \times n$ matrix of fuzzy numbers, $m \leq n, b$ a column vector of fuzzy numbers and $c \in \Re^n$.
The imprecision of the coefficients can be modeled by means of fuzzy numbers. On the other hand, the decision maker tolerates violation in the accomplishment of the constraints.

The auxiliary problem to solve (4) is:

Maximize $c.x$
Subject to $A.x <^f b + t.(1-\alpha)$ (5)
$x \geq 0,\ \alpha \in (0;1]$

The relation $<^f$ is any the decision maker chooses. According to the kind of relation $<^f$, which we assume, different models of conventional linear programming problems will be obtained.[2]

3.1. First approach

We consider that the unit price and the variable unit cost for each product a firm produces are represented by real numbers, in other words, they are well-known values, while the total fixed costs are imprecise.
The crisp model we have proposed in 2. turns into a fuzzy linear programming model with crisp objective

function, crisp technological coefficients and imprecise resources.
Since break-even condition (1) will have some imprecise coefficients, we cannot state the equality remains valid.
For this reason equation (1) will be replaced by the inequalities:

$$\sum_{i=1}^{n}(p_i - w_i).q_i \gtrsim f \qquad (6)$$

$$\sum_{i=1}^{n}(p_i - w_i).q_i \leq m \qquad (7)$$

(6) marginal contributions must cover at least total fixed costs.
(7) m is the optimal marginal contribution, solution to a maximizing crisp linear programming problem that includes the constraints of the original one. It is boundary to (6).

We will name (6) *approximate break-even condition*.
Generalizing our approach in 2., we have a fuzzy linear programming model with crisp objective function and fuzzy constraints.

Minimize: $I = \sum_{i=1}^{n} p_i.q_i$

Subject to:

$$\sum_{i=1}^{n} m_i.q_i \gtrsim f \qquad (8)$$

$$\sum_{i=1}^{n} m_i.q_i \leq m \qquad (9)$$

$$A.q_i :: B \qquad (10)$$

$$q_i \geq 0 \qquad (11)$$

(8) *approximate break – even condition*, where $m_i = p_i - w_i$ is the unit marginal contribution of the product i.
(9) boundary to fuzzy inequality (8)
(10) crisp or fuzzy restrictions
(11) non–negativity of the decision variables

The analysis of this situation is very useful to the firm because it allows to find the minimum sale income so that marginal contributions cover at least the total fixed costs.
There are several methods to solve the fuzzy linear programming model with fuzzy constraints such as the ones proposed by Delgado, M., Verdegay, J.L. and Vila, M.A. [2], Tanaka, H., Okuda, T. and Asai, K. [7], Verdegay, J.L. [9].

3.1.1. Study of a Case

A nursery school in Buenos Aires offers full-time service, part-time service (either morning or afternoon) and extra hours (two at the most) that can be added to the part –time services.
The maximum capacity of the school is 36 pupils per turn. It is also known that no more than 20% of the children stay at school an extra hour, no more than 5% of the pupils stay at school two extra hours and at the most 30% of the children are full–time pupils.
Meals are provided by a catering service firm. Lunch unit cost is $14 per month and breakfast/afternoon snack unit cost is $8. These are the only unit variable costs.
It is required to calculate the number of services of each type needed to cover at least the total fixed and variable costs.

We consider the following decision variables,
q_1: full-time service,
q_2: part-time service (morning),
q_3: part-time service (afternoon),
q_4: one extra hour,
q_5: two extra hours
The fees applied to each service offered are
$p_1 = 150 \quad p_2 = 90 \quad p_3 = 90 \quad p_4 = 20 \quad p_5 = 40$
In all the cases the price includes lunch, breakfast and/or afternoon snack.
Using the above information, variable unit costs are
$w_1 = 30,\ w_2 = 22,\ w_3 = 22,\ w_4 = 0,\ w_5 = 0$
The corresponding unit marginal contributions are
$m_1 = 120,\ m_2 = 68,\ m_3 = 68,\ m_4 = 20,\ m_5 = 40$ The total fixed costs are no less than 2862 and no more than 3583.
Replacing in the first approach we have the following fuzzy linear programming problem,

Minimize
$I = 150.q_1 + 90.q_2 + 90.q_3 + 20.q_4 + 40.q_5$
Subject to

$$q_i \in Z \quad 1 \leq i \leq 5 \qquad (12)$$

$$q_i \geq 0 \quad 1 \leq i \leq 5 \qquad (13)$$

$$q_1 + q_2 \leq 36 \qquad (14)$$

$$q_1 + q_3 \leq 36 \qquad (15)$$

$$q_4 \leq (q_2 + q_3).0,2 \qquad (16)$$

$$q_5 \leq (q_2 + q_3).0,05 \qquad (17)$$

$q_1 \le (q_1 + q_2 + q_3).0,3$ (18)

$120.q_1 + 68.q_2 + 68.q_3 + 20.q_4 + 40.q_5 \gtrsim 3583$ (19)

$120.q_1 + 68.q_2 + 68.q_3 + 20.q_4 + 40.q_5 \le 5296$ (20)

(12) additional constraint: the decision variables must be integer numbers
(13) non-negativity of the decision variables
(14, 15) the maximum capacity of the school is 36 pupils per turn
(16) no more than 20% of the children stay at school an extra hour
(17) no more than 5% of the children stay at school two extra hours
(18) at the most 30% of the children are full-time pupils
(19) the maximum tolerance for f is $t = 721$
(20) its right hand side is the optimal solution to the crisp linear programming problem of maximizing the marginal contribution subject to the constraints (12,18).

We use the general model for fuzzy linear programming proposed by Delgado, M., Verdegay, J.L. and Vila, M.A. [2] in the special case when only the coefficients are fuzzy. The problem turns into the following classic parametric linear programming problem with crisp solution.

Minimize
$I = 150.q_1 + 90.q_2 + 90.q_3 + 20.q_4 + 40.q_5$
Subject to

$$q_i \in Z \quad 1 \le i \le 5$$
$$q_i \ge 0 \quad 1 \le i \le 5$$
$$q_1 + q_2 \le 36$$
$$q_1 + q_3 \le 36$$
$$q_4 \le (q_2 + q_3).0,2$$
$$q_5 \le (q_2 + q_3).0,05$$
$$q_1 \le (q_1 + q_2 + q_3).0,3$$

The fuzzy inequality (17) is replaced by the crisp inequality

$$120.q_1 + 68.q_2 + 68.q_3 + 20.q_4 + 40.q_5 \ge 3583 - 721.\beta$$
$$0 \le \beta \le 1$$
$$120.q_1 + 68.q_2 + 68.q_3 + 20.q_4 + 40.q_5 \le 5296$$

A specialized software application for integer linear programming is used to solve this problem. The solution for some values of β is shown in Table 1.

From the table we see that it is an *approximate break-even* situation.

Table 1: Solution of the case

β	q_1	q_2+q_3	q_4	q_5	Income	Profit
0	11	31	6	1	4600	5.0
.2	12	28	5	0	4420	5.2
.4	11	27	5	1	4220	1.4
.6	10	27	4	1	4050	5.6
.8	10	25	4	1	3870	13.8
1	9	25	5	0	3700	18.0

3.2. Second approach

Let us consider the case where, according to the available information, the unit price for each product a firm produces is a crisp number, the variable cost for each product i is estimated by the triangular fuzzy number $\tilde{W}_i = (w_{i1}, w_{i2}, w_{i3})$ and the triangular fuzzy number $\tilde{F} = (f_1, f_2, f_3)$ represents the total fixed costs.

The crisp model we proposed in 2. turns into a fuzzy linear programming model with crisp objective function, fuzzy technological coefficients and fuzzy resources.

In this situation it is also necessary to replace break-even condition (1) by an *approximate break- even condition*.

Generalizing our approach in 2., the problem of finding the *approximate break-even* when marginal contributions and fixed costs are triangular fuzzy numbers is expressed by:

Minimize: $I = \sum_{i=1}^{n} p_i.q_i$

Subject to: $\sum_{i=1}^{n} \tilde{M}_i.q_i \ge f_3$ (21)

$\sum_{i=1}^{n} \tilde{M}_i.q_i \le m$ (22)

$$A.q_i :: B$$
$$q_i \ge 0$$

(21) *approximate break – even condition*, where $\tilde{M}_i = p_i - \tilde{W}_i$ is the unit marginal contribution of the product i
(22) m is the highest marginal contribution, optimal solution to a maximizing fuzzy linear programming problem that includes the constraints of the original problem. As the

information is given by triangular fuzzy numbers, it can be solved by using a multiple objective linear programming method such as the one proposed by Young–Jou Lai and Ching-Lai Hwang [6].

≥ is a fuzzy ranking method.

Note that the right hand side of inequality (21) is f_3 because the highest total fixed costs must be covered in every case.

Since there are many methods to order fuzzy numbers, the choice of one of them will generate a method of fuzzy linear programming to solve this model. Among these methods we can mention the ones developed by Delgado, M., Verdegay, J.L. and Vila, M.A. [2], Tanaka H., Ichihashi H. and Asai K [8], Klir, J.G and Yuan, B. [5].

This model would be useful to deal with the case of study if meals were cooked at school. In this situation, variable costs would be imprecise and triangular fuzzy numbers would be suitable for expressing their vagueness.

4. Remarks

There are situations where the available information is fuzzy and the break-even point cannot be found using the classic methods. After our analysis we conclude that these cases are satisfactorily solved using the *approximate break-even condition*, which allows to calculate the number of services or products of each type that must be sold to cover at least the total fixed costs.

By using fuzzy linear programming it is quite simple to deal with the problem of calculating break-even under conditions of uncertainty. Data is processed using a specialized software application for real or integer linear programming.

In the second approach marginal contributions and total fixed costs can be expressed by any type of fuzzy number. In every case the right-hand side of inequality (21) is the supremum of the support of the total fixed costs.

We could consider a third approach where unit prices are imprecise. In this case it would be useful to study profit.

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