Fuzzy Relation Equations and Approximate Solutions. Some Recent Results

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Abstract

For theoretical fuzzy control it is a well known strategy to transform a system of control rules into a system of relation equations. Because these systems of relation equations are not always solvable, solvability criteria and approximate solutions have been discussed.

We reconsider some of the results in this field and extend them using more recent results on t-norm based fuzzy logics and on approximation processes.

1 Fuzzy control and relation equations

A fuzzy controller usually is determined by a finite list

$$\text{if } \alpha \text{ is } A_i, \text{ then } \beta \text{ is } B_i, \quad i = 1, \dots, n\,, \tag{1}$$

of linguistic control rules, which are supposed to be designed to describe some control procedure with input variable α and output variable β.

The standard understanding, originating from [15], is that such a fuzzy controller should be realized by a fuzzy relation R which connects fuzzy input information A with fuzzy output information B via the *compositional rule of inference* CRI

$$B = A \circ R = \{y \,\|\, \exists x(A(x) \& R(x, y)\}\,. \tag{2}$$

Each of the control rules from (1) determines the single equation $B_i = A_i \circ R$. Therefore the system (1) of linguistic control rules is to be transformed into a system of fuzzy relation equations

$$A_i \circ R = B_i, \qquad \text{for } i = 1, \dots, n\,. \tag{3}$$

Related to the system (3) of relation equations, one considers the fuzzy relation

$$\hat{R} = \bigcap_{i=1}^{n} \{(x, y) \,\|\, A_i(x) \to B_i(y)\}\,. \tag{4}$$

As is well known, and explained e.g. in [2], one has the following result.

Theorem 1 *The system* (3) *of relation equations is solvable iff* $\hat{R}$ *is a solution of* (3)*.*

Besides the approach toward systems of linguistic control rules (1) via the solvability of the systems (3) of relation equations, and via their maximal (possible) solutions (4), one has the competing approach of Mamdani/Assilian [9]. This approach reads the list (1) of linguistic control rules as a (rough) description of a fuzzy function determined by the fuzzy relation

$$R_{\text{MA}} = \bigcup_{i=1}^{n} (A_i \times_{\boldsymbol{t}} B_i) \tag{5}$$

and the standard mapping procedure (2) for fuzzy subsets A of the input space.

The comparison of both approaches poses the problem under which conditions the fuzzy relation R_{MA} is a solution of the system (3). A characterization is given in [6]:

Theorem 2 *Let all the input sets* A_i *be normal. Then the fuzzy relation* R_{MA} *is a solution of the system* (3) *of fuzzy relation equations iff for all* $i, j = 1, \dots, n$ *one has*

$$\models \exists x(A_i(x) \& A_j(x)) \to B_i \equiv^*_{\boldsymbol{t}} B_j\,. \tag{6}$$

Here the t-norm based graded identity relation $\equiv^*_{\boldsymbol{t}}$ is defined as

$$A \equiv^*_{\boldsymbol{t}} B = A \mathrel{\tilde{\subseteq}_{\boldsymbol{t}}} B \wedge B \mathrel{\tilde{\subseteq}_{\boldsymbol{t}}} A\,, \tag{7}$$

using the graded inclusion relation

$$A \mathrel{\tilde{\subseteq}_{\boldsymbol{t}}} B = \forall x(A(x) \to B(x))\,.$$

The condition (6) states a kind of functionality of the list (1) of control rules because (6) can be rewritten as

$$\models A_i \cap_{\boldsymbol{t}} A_j \not\equiv^*_{\boldsymbol{t}} \emptyset \to B_i \equiv^*_{\boldsymbol{t}} B_j\,. \tag{8}$$

And this indeed can be understood as a fuzzification of the idea that "if A_i and A_j coincide to some degree, than also B_i and B_j should coincide to a certain degree".

Looking back at condition (6) or (8), one recognizes that it is symmetric in i, j. This give a slightly sharpened version of KLAWONN's solvability criterion (6).

Corollary 3 *Let all the input sets A_i be normal. Then the fuzzy relation R_{MA} is a solution of the system (3) of fuzzy relation equations iff for all $i, j = 1, \ldots, n$ one has*

$$\models A_i \cap_{\boldsymbol{t}} A_j \not\equiv^*_{\boldsymbol{t}} \emptyset \to B_i \mathrel{\tilde{\subseteq}_{\boldsymbol{t}}} B_j\,.$$

In general, the determination of $\widehat{R}$ as well as that of R_{MA} amounts to form *pseudo-solutions*, sometimes also called *approximate solutions*.

2 Asking for lower and upper approximations

An old approach from [2] has been to subdivide the problem whether a fuzzy relation R is a solution of the system (3) into the *subset property*

$$A_i \circ R \subseteq B_i, \qquad \text{for } i = 1, \ldots, n\,, \tag{9}$$

and the *superset property*

$$A_i \circ R \supseteq B_i, \qquad \text{for } i = 1, \ldots, n\,, \tag{10}$$

Particularly for the relation R_{MA} quite natural sufficient conditions for the superset property have been given, but only rather strong ones for the subset property:

Proposition 4 *If all input sets A_i are normal then R_{MA} has the superset property.*

Proposition 5 *If all input sets are pairwise $\cap_{\boldsymbol{t}}$-disjoint, then R_{MA} has the subset property.*

It is also of interest to ask for conditions under which the relation $\hat{R}$ satisfies these properties. For the subset property there is a nice answer.

Proposition 6 *$\hat{R}$ has the subset property.*

Proposition 7 *If the input set A_k is normal then*

$$A_k \circ \hat{R} \subseteq B_k \subseteq A_k \circ R_{\mathrm{MA}}\,. \tag{11}$$

Proposition 8 *If all the input sets A_i of the system (3) are normal and if one also has $R_{\mathrm{MA}} \subseteq \hat{R}$, then system (3) is solvable, and R_{MA} is a solution.*

Corollary 9 *Assume the normality of all the input sets A_i. Then to have $A_i \circ R_{\mathrm{MA}} \subseteq A_i \circ \hat{R}$ for all indices i is equivalent to the fact that R_{MA} is a solution of system (3), and hence equivalent to criterion (8).*

Corollary 10 *The system (3) of relation equations is solvable iff the relation $\hat{R}$ has the superset property.*

Corollary 11 *For continuous t-norms $\boldsymbol{t}$ a necessary condition for the superset property of $\hat{R}$ is that $\mathsf{hgt}\,(B_k) \leq \mathsf{hgt}\,(A_k)$ holds for all input-output pairs (A_k, B_k) of the system (3).*

What actually remain as **open problems** is to find conditions which are *equivalent* to (10), or at least *sufficient* for (10).

3 Iterating pseudo-solution strategies

Another idea is to find new pseudo-solution strategies by a kind of *iteration* of the basic pseudo-solution strategies. This is done in the way, that for the "next step" in such an iteration process the system (3) is changed such that its output sets become the real output of the former iteration step.

To indicate the dependence of the pseudo-solutions R_{MA} and $\hat{R}$ from the (input and) output data, the "original" pseudo-solutions with the input-output data (A_i, B_i) shall be written

$$R_{\mathrm{MA}}[B_k] \text{ for } R_{\mathrm{MA}}, \quad \hat{R}[B_k] \text{ for } \hat{R}.$$

We mention here only the (perhaps modified) output data explicitly because the input data shall be the same in all cases we are going to discuss.

Now one has the following results, cf. [5].

Proposition 12 *For any fuzzy relation S one has for all i:*

$$A_i \circ \hat{R}[A_k \circ S] = A_i \circ S.$$

Hence it does not give a new pseudo-solution if one iterates the S-solution strategy after some (other) pseudo-solution. The situation changes if one uses the MA-solution strategy after an other pseudo-solution, cf. [5].

Theorem 13 *One has always*

$$A_i \circ \hat{R}[B_k] \subseteq A_i \circ R_{\mathrm{MA}}[A_k \circ \hat{R}[B_k]],$$
$$A_i \circ R_{\mathrm{MA}}[A_k \circ \hat{R}[B_k]] \subseteq A_i \circ R_{\mathrm{MA}}[B_k].$$

Thus the iterated relation $R_{\mathrm{MA}}[A_k \circ \hat{R}]$ is a *better* pseudo-solution as each one of R_{MA} and $\hat{R}$.

4 Introducing the solvability degree

For a more detailed discussion of the solvability behavior solvability degrees have been introduced in [1], and in more detail explained e.g. in [3]. We need the (global) *solvability degree*

$$\xi = [\![\exists X \prod_{i=1}^{n}(A_k \circ X \equiv_{\boldsymbol{t}} B_i)]\!],$$

for the system (3), and for any fuzzy relation R the *solution degree*

$$\delta(R) = [\![\prod_{i=1}^{n}(A_i \circ R \equiv_{\boldsymbol{t}} B_i)]\!].$$

As an additional notation we use $z(u)$ for the largest $\boldsymbol{t}$-idempotent below u, as e.g. in [11].

The following result had been proved in [1] and again discussed in [2]:

Theorem 14 $\xi^n \leq \delta(\hat{R}) \leq \xi$.

Of course, the n-th power here is the iteration of the t-norm $\boldsymbol{t}$. So one gets immediately

Corollary 15 $\delta(\hat{R})^n \leq \xi^n \leq \delta(\hat{R})$.

Furthermore one has

Proposition 16 *For each continuous t-norm $\boldsymbol{t}$ and each $1 \leq n \in \mathbb{N}$ there exists n-th roots.*

Corollary 17 *For t-norms which have n-th roots one has the inequalities*

$$\delta(\hat{R}) \leq \xi \leq \sqrt[n]{\delta(\hat{R})}.$$

Related to our discussions concerning KLAWONN's criterion for the solution property of R_{MA}, it is of interest also to determine the solution degree of the relation R_{MA}.

Proposition 18 *If all input sets A_i are normal then*

$$\delta(R_{\mathrm{MA}}) = [\![\prod_i \bigwedge_j (A_i \cap_{\boldsymbol{t}} A_j \not\equiv \emptyset \rightarrow B_i \subseteq\!\!\equiv_{\boldsymbol{t}} B_j)]\!].$$

This is obviously a generalization of the KLAWONN criterion, particularly of the form we gave it in Corollary 3.

Now we look at the solution degree of the pseudo-solution $R_{\mathrm{MA}}[A_k \circ \hat{R}[B_k]]$. And we find another result which indicates that $R_{\mathrm{MA}}[A_k \circ \hat{R}[B_k]]$ is (sometimes) as good a pseudo-solution as R_{MA}.

Proposition 19 *For normal input sets A_i with*

$$\models B_i \subseteq\!\!\equiv_{\boldsymbol{t}} B_j \rightarrow A_i \circ \hat{R} \subseteq\!\!\equiv_{\boldsymbol{t}} A_j \circ \hat{R},$$

one has

$$\delta(R_{\mathrm{MA}}) \leq \delta(R_{\mathrm{MA}}[A_k \circ \hat{R}[B_k]]).$$

5 CRI is an approximation strategy

In fuzzy control the true object which has to be determined, viz. the control function Φ, additionally is described only roughly, i.e. given only by its behavior in some (fuzzy) points of the state space. The list (1) of control rules just means

$$\Phi^*(A_i) = B_i, \quad i = 1, \ldots, n \qquad (12)$$

for a suitable "fuzzified" version $\Phi^* : I\!F(\mathcal{X}) \to I\!F(\mathcal{Y})$ of the control function $\Phi : \mathcal{X} \to \mathcal{Y}$.

And the additional approximation idea explained in ZADEH's CRI is that one likes to approximate Φ^* by a fuzzy function $\Psi^* : I\!F(\mathcal{X}) \to I\!F(\mathcal{Y})$ which is determined for all $A \in I\!F(\mathcal{X})$ by the equation

$$\Psi^*(A) = A \circ R \tag{13}$$

which refers to some suitable fuzzy relation R and understands $\circ$ as sup-t-composition.

Formally this means that the equations (12) become transformed into the system (3) which has to be solved w.r.t. the fuzzy relation R.

This approximation idea fits well with the fact that one often is satisfied with pseudo-solutions of (3), and particularly with the MA-pseudo-solution R_{MA} of MAMDANI/ASSILIAN [9], or the S-pseudo-solution $\widehat{R}$ of SANCHEZ [13].

Neither R_{MA} nor $\widehat{R}$ needs to be a solution of (3), however both these pseudo-solutions determine approximations Ψ^* to the (fuzzified) control function Φ^*.

6 Approximate solutions of fuzzy relation equations

The author of this paper used in previous papers, e.g. in [2, 5], the notion of approximate solution only naively in the previously explained sense of a fuzzy relation which roughly describes the intended control behavior which some list (1) of (linguistic) control rules describes.

A precise qualitative definition of a notion of approximate solution was given by WU [14] and used e.g. by KLIR/YUAN [7, 8]. In this approach an approximate solution $\widetilde{R}$ of (3) is a fuzzy relation which satisfies the following two conditions:

i. There are fuzzy sets ${A_i}', {B_i}'$ such that for all $i = 1, \ldots, n$ one has $A_i \subseteq {A_i}'$ and ${B_i}' \subseteq B_i$ as well as ${A_i}' \circ \widetilde{R} = {B_i}'$.

ii. If there exist fuzzy sets ${A_i}^*, {B_i}^*$ for $i = 1, \ldots, n$ and a fuzzy relation R^* such that ${A_i}^* \circ R^* = {B_i}^*$ and $A_i \subseteq {A_i}^* \subseteq {A_i}'$ as well as ${B_i}' \subseteq {B_i}^* \subseteq B_i$ for all $i = 1, \ldots, n$ then one has ${A_i}^* = {A_i}'$ and ${B_i}^* = {B_i}'$ for all $i = 1, \ldots, n$.

These conditions formalize the ideas that (i) an approximate solution $\widetilde{R}$ should be a solution of a system of relation equations with input-output data $({A_i}', {B_i}')$ which may (slightly) differ from the original input-output data (A_i, B_i) and which (ii) has the additional property that no system of relational equations with input-output data which "strongly better" approximate the original ones is solvable.

7 Some generalizations of WU's approach

It is obvious that the two conditions (i), (ii) of WU are independent.

What is, however, not obvious at all – and even rather arbitrary – is that condition (i) also says that the approximating input-output data $({A_i}', {B_i}')$ should approximate the original input data *from above*[1] and the original output data *from below*. To overcome this artificial restriction we redefine the crucial notion of approximate solution here in the following way.

Before we coin the name of an *approximating system* for (3) and understand by it any system

$$C_i \circ R = D_i, \qquad i = 1, \ldots, n \tag{14}$$

of relation equations with the same number of equations.

Definition 1 *A* ul-approximate solution *of a system* (3) *of relation equations is a solution of a* ul-approximating system *for* (3)*, i.e. of an approximating system* (14) *for* (3) *which satisfies*

$$A_i \subseteq C_i \ \textit{ and } \ B_i \supseteq D_i, \quad \textit{for } i = 1, \ldots, n\,. \tag{15}$$

An l*-approximate solution *of a system* (3) *of relation equations is a solution of an* l*-approximating system (14) *for* (3)*, i.e. of a system which satisfies*

$$A_i \supseteq C_i \ \textit{ and } \ B_i = D_i, \quad \textit{for } i = 1, \ldots, n\,. \tag{16}$$

In a similar way one may define the notions of lu-approximate solution, of ll-approximate solution,

[1] The ordering we refer to here and later on in this discussion is the usual inclusion for fuzzy sets.

of uu-approximate solution, of u*-approximate solution, of *l-approximate solution, and of *u-approximate solution.

Corollary 20 *(i) Each *l-approximate solution of* (3) *is also an ul-approximate solution and an ll-approximate solution of* (3)*.*

(ii) Each u-approximate solution of* (3) *is also an ul-approximate solution and an uu-approximate solution of* (3)*.*

Proposition 21 *For each system* (3) *of relation equations its S-pseudo-solution* $\widehat{R}$ *is an *l-approximate solution.*

This generalizes a result of Klir/Yuan [7].

Proposition 22 *For each system* (3) *of relation equations with normal input data its MA-pseudo-solution* R_{MA} *is an *u-approximate solution.*

Together with Corollary 20 these two Propositions say that each system of relation equations has approximate solutions of any one of the types we introduced in this section.

8 Optimality of approximate solutions

All the previous results do not give any information about some kind of "quality" of the approximate solutions or the approximating systems.

Definition 2 *An approximate solution* $\widetilde{R}$ *of a system* (3) *is called* optimal *iff there does not exist a solvable system* $R''C_i' = D_i'$ *of relation equations whose input-output data* (C_i', D_i') *approximate the original input-output data of* (3) *strongly better than the input-output data* (C_i, D_i) *of the system which determines* $\widetilde{R}$*.*

Proposition 23 *If an *l-approximate solution* $\widetilde{R}$ *is optimal, then it is also optimal as a ul-approximate and as an ll-approximate solution.*

Of course, similar results holds true also for (l*-, u*- and) *u-approximate solutions.

9 Some optimality results

The problem arises immediately whether the two standard pseudo-solutions $\widehat{R}$ and R_{MA} are optimal (inclusion based) approximate solutions. For the S-pseudo-solution $\widehat{R}$ as an ul-approximate solution this optimality was shown in [7, 8].

$\widehat{R}$ is even an $\subseteq$-optimal *l-approximate solution. The proofs of this and the other results mentioned here can be found in [12].

Proposition 24 *The fuzzy relation* $\widehat{R}$ *is always an* $\subseteq$*-optimal *l-approximate solution.*

For the MA-pseudo-solution the situation is different.

Proposition 25 *There exist systems* (3) *for which their MA-pseudo-solution* R_{MA} *is an *u-approximate solution which is not optimal.*

Definition 3 *Call a system* (3) MA-solvable *iff* R_{MA} *is a solution of this system.*

Proposition 26 *If* (3) *has an MA-solvable *u-approximating system* $A_i \circ R = B_i^*$ *such that for the MA-pseudo-solution* R_{MA} *of* (3) *one has*

$$B_i \subseteq B_i^* \subseteq A_i \circ R_{\mathrm{MA}}\,, \qquad i = 1, \dots, n\,,$$

then R_{MA} *solves this *u-approximating system.*

Corollary 27 *If all input sets of* (3) *are normal then the smallest MA-solvable *u-supersystem of* (3) *has input-output data* $(A_i, A_i \circ R_{\mathrm{MA}})$*.*

Corollary 28 *Let* $\widehat{R}$ *be the S-pseudo-solution of* (3)*, let be* $\widehat{B_i} = A_i \circ \widehat{R}$*, and suppose that the system*

$$A_i \circ R = \widehat{B_i}\,, \qquad i = 1, \dots, n\,, \tag{17}$$

is MA-solvable. Then $R_{\mathrm{MA}}[A_k \circ \widehat{R}[B_k]]$ *is an optimal *l-approximate solution of* (3)*.*

10 Modifying the approximation behavior via t-norm changes

For t-norms $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ their ordering $\leqq$ as functions given by

$$\boldsymbol{t}_1 \leqq \boldsymbol{t}_2 \quad \text{iff} \quad \text{always } \boldsymbol{t}_1(x,y) \le \boldsymbol{t}_2(x,y)\,.$$

Call $\boldsymbol{t}_1$ *weaker* than $\boldsymbol{t}_2$ iff $\boldsymbol{t}_1 \leqq \boldsymbol{t}_2$ holds true.

For t-norms $\boldsymbol{t}_k$ for $k = 1, 2$ write $\&_k$ and $\rightarrow_k$ for the connectives $\&_{\boldsymbol{t}_k}, \rightarrow_{\boldsymbol{t}_k}$. Write also

$$\widehat{R}^k \;\text{ for }\; \widehat{R}^{(\boldsymbol{t}_k)}\,, \qquad {R^k}_{\mathrm{MA}} \;\text{ for }\; {R^{(\boldsymbol{t}_k)}}_{\mathrm{MA}}\,.$$

Proposition 29 *For t-norms $\boldsymbol{t}_1, \boldsymbol{t}_2$ one has for the pseudo-solutions ${R^k}_{\mathrm{MA}}$ and $\widehat{R}^k$ of a system (3) of relation equations*

$$\boldsymbol{t}_1 \leqq \boldsymbol{t}_2 \quad \Rightarrow \quad {R^1}_{\mathrm{MA}} \subseteq {R^2}_{\mathrm{MA}}\,, \qquad (18)$$
$$\boldsymbol{t}_1 \leqq \boldsymbol{t}_2 \quad \Rightarrow \quad \widehat{R}^2 \subseteq \widehat{R}^1\,. \qquad (19)$$

Proposition 30 *For t-norms $\boldsymbol{t}_1 \leqq \boldsymbol{t}_2$ one has for any system (3) and for all t-norms $\boldsymbol{t}$*

$$A_i \circ \widehat{R}^2 \subseteq A_i \circ \widehat{R}^1 \subseteq B_i \subseteq A_i \circ R^1_{\mathrm{MA}} \subseteq A_i \circ R^2_{\mathrm{MA}}\,.$$

with $\circ$ denoting sup-$\boldsymbol{t}$-composition.

This result means that the two standard pseudo-solutions w.r.t. the weaker t-norm give better approximate solutions for the system (3) of relation equations.

To evaluate this result, the reader has to have in mind that in these inclusions the $\boldsymbol{t}_k$-related pseudo-solutions are used together with sup-$\boldsymbol{t}$-composition for an arbitrary t-norm $\boldsymbol{t}$. The standard situation is, however, to consider $\boldsymbol{t}_k$-related pseudo-solutions together with sup-$\boldsymbol{t}_k$-composition. For this standard situation we get

Corollary 31 *For t-norms $\boldsymbol{t}_1 \leqq \boldsymbol{t}_2$ one has for any system (3)*

$$B_i \subseteq A_i \circ_{\boldsymbol{t}_1} R^1_{\mathrm{MA}} \subseteq A_i \circ_{\boldsymbol{t}_2} R^2_{\mathrm{MA}}\,,$$

which means that weaker t-norms provide better MA-pseudo-solvability.

A similar result w.r.t. the S-pseudo-solutions is unknown: and it is an open problem whether such a result may hold at all.

For the particular case of MA-solvable systems of relation equations this result gives immediately

Corollary 32 *If a system* (3) *of relation equations is MA-solvable w.r.t. some t-norm $\boldsymbol{t}$ then it is also MA-solvable w.r.t. every weaker t-norm.*

And generally one has also the following evaluation for the most preferred choice of t-norms.

Corollary 33 *For the largest t-norm* min *the MA-pseudo-solution R_{MA} gives the worst possible MA-approximation quality among all t-norms.*

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