

Aggregating non-finite families of T-transitive relations

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Abstract

Aggregation of non-finite families of transitive relations is studied and applied to calculate the degrees of inclusion and similarity of fuzzy quantities (fuzzy subsets of an interval of the real line).

Keywords: Indistinguishability operator, similarity, fuzzy equivalence relations, fuzzy preorder, T-transitive fuzzy relation, aggregation, means.

1 Introduction

Transitivity is one of the most important properties that can be required to a fuzzy relation. The most common used transitive fuzzy relations are indistinguishability operators and fuzzy preorders. There is a lot of work around these two concepts and have been proved useful tools both in the theoretical aspects of fuzzy logic as well as in their applications ([2],[9]).

In many situations, there can be more than one such relations defined on a universe. In this cases we may need to aggregate them. The usual way to do it is calculating the Minimum (or Infimum) of them. Although this has a very clear interpretation in fuzzy logic [3] since the Infimum is used to model the universal fuzzy quantifier \forall , it leads many times to undesirable results in applications. In this cases we need to use other ways to aggregate fuzzy relations. Some basic ideas about this topic can be found in [4],[5],[6]. A general overview about preservation of transitivity, in the discret case, can be found in [7].

After some preliminaries, section 3 will be focused on the use of means to aggregate transitive relations.

In Section 4 we will extend our results to aggregate a non-finite family of transitive relations. We will apply the obtained results to calculate the degrees of inclusion and similarity of fuzzy quantities (fuzzy subsets of an interval of the real line).

2 Preliminaries

In this Section we will recall some well known definitions and properties. Throughout the paper T will denote a continuous t-norm, although most results remain valid assuming only left continuity for T .

Definition 2.1. A fuzzy relation R on a set X is T -transitive if and only if for all x, y, z of X satisfies

$$T(R(x, y), R(y, z)) \leq R(x, z)$$

Among all T -transitive fuzzy relations, T -indistinguishability operators and fuzzy T -preorders are the most important ones.

Definition 2.2. A fuzzy relation E on a set X is a T -indistinguishability operator if and only if for all x, y, z of X satisfies the following properties

- (i) $E(x, x) = 1$ (*Reflexivity*)
- (ii) $E(x, y) = E(y, x)$ (*Symmetry*)
- (iii) $T(E(x, y), E(y, z)) \leq E(x, z)$ (*Transitivity*)

Definition 2.3. A fuzzy relation P on a set X is a fuzzy T -preorder if and only if for all x, y, z of X satisfies the following properties

- (i) $P(x, x) = 1$ (*Reflexivity*)
- (ii) $T(P(x, y), P(y, z)) \leq P(x, z)$ (*Transitivity*)

Definition 2.4. The quasi-inverse or residuation of a t-norm T is defined $\forall x, y \in [0, 1]$ by

$$\hat{T}(x|y) = \sup\{\alpha \in [0, 1] | T(\alpha, x) \leq y\}.$$

The bi-residuation $\overleftrightarrow{T}(x, y)$ is defined by

$$\overleftrightarrow{T}(x, y) = T(\hat{T}(x|y), \hat{T}(y|x))$$

Proposition 2.5. [8] Given a fuzzy set μ of X , the fuzzy relation E on X defined for all x, y of X by $E_\mu(x, y) = \overleftrightarrow{T}(\mu(x), \mu(y))$ is a T -indistinguishability operator.

Proposition 2.6. [8] Given a fuzzy set μ of X , the fuzzy relation P on X defined for all x, y of X by $P_\mu(x, y) = \hat{T}(\mu(x) | \mu(y))$ is a fuzzy T -preorder.

3 Aggregating transitive fuzzy relations

The most common way to put together a family of T -transitive fuzzy relations is calculating their infimum.

Proposition 3.1. [8] Let $(R_i)_{i \in I}$ be a family of T -transitive fuzzy relations on a set X . The relation defined for all $x, y \in X$ by

$$R(x, y) = \inf_{i \in I} R_i(x, y)$$

is a T -transitive fuzzy relation on X .

Nevertheless, in many situations this way of aggregating fuzzy relations leads to undesirable results since the Infimum only takes into account the smaller value for every couple and forgets or loses the information of the other ones.

A possibility to soften the previous proposition is replacing the Infimum by the t-norm T as the following proposition shows.

Proposition 3.2. Let R_1, R_2, \dots, R_n be n T -transitive fuzzy relations on a set X . The relation R defined for all $x, y \in X$ by

$$R(x, y) = T(R_1(x, y), R_2(x, y), \dots, R_n(x, y))$$

is a T -transitive fuzzy relation on X .

The previous proposition has some advantages with respect to Proposition 3.1. For example, if we want to aggregate two relations and one is twice as important as the other, we can count the most important twice. But on the other hand, it can produce relations with very small values and if T is a non-strict archimedean t-norm, most of them will probably be 0.

So, more general ways to aggregate transitive relations are needed.

Definition 3.3. A (bisymmetric) mean is a map $m : [0, 1]^2 \rightarrow [0, 1]$ such that for all $x, y, z, t \in [0, 1]$ satisfies

- (i) $\text{Min}(x, y) \leq m(x, y) \leq \text{Max}(x, y)$
- (ii) $m(x, y) = m(y, x)$
- (iii) m is non decreasing on each variable
- (iv) $m(m(x, y), m(z, t)) = m(m(x, z), m(y, t))$

Proposition 3.4. [1] m is a continuous mean in $[0, 1]$ if and only there exists a continuous strict monotonic map $t : [0, 1] \rightarrow R$ such that for all $x, y \in [0, 1]$

$$m(x, y) = t^{-1}\left(\frac{t(x) + t(y)}{2}\right).$$

Since the additive generator of an archimedean t-norm is a decreasing map $t : [0, 1] \rightarrow R$, this gives a way to generate a mean m_t associated to an archimedean t-norm.

Lemma 3.5. Let $t, t' : [0, 1] \rightarrow R$ be two continuous strict monotonic maps and $m_t, m_{t'}$ the means generated by t and t' according to Proposition 3.4. respectively.

- (a) If t, t' differ only by an additive constant, then $m_t = m_{t'}$.
- (b) If t, t' differ only by a multiplicative positive constant, then $m_t = m_{t'}$.
- (c) $m_t = m_{-t}$.

In particular, if t and t' are two additive or multiplicative generators of T , then the means generated by t and t' coincide (i.e.: $m_t = m_{t'}$).

Proposition 3.6. The map assigning to every archimedean t-norm T with generator t the mean m_t generated by t is a bijection between the set

of archimedean t-norms and the set of continuous means.

It is straightforward to extend the previous proposition to more than two variables and to weighted means.

Proposition 3.7. Let T be an archimedean t-norm with generator t and R_1, R_2, \dots, R_n a family of T -transitive fuzzy relations on a set X . The fuzzy relation R on X defined $\forall x, y \in X$ by

$$R(x, y) = m_t(R_1(x, y), R_2(x, y), \dots, R_n(x, y))$$

is T -transitive.

So means are operators preserving T -transitivity and from a family of relations they give as a result a relation with values between the greatest and the lowest ones of the family. Therefore they seem a good way to aggregate T -transitive relations.

4 Aggregating a non-finite number of relations

In some cases we have to aggregate a non-finite number of relations. Let T be an archimedean t-norm with additive generator t and let us suppose that we have a family of T -transitive relations $(R_i)_{i \in [a, b]}$ on a set X with the indices in the interval $[a, b]$ of the real line and that for every couple (x, y) of X^2 the map $f_{(x, y)} : [a, b] \rightarrow R$ defined by $f_{(x, y)}(i) = R_i(x, y)$ is integrable in some sense.

Definition 4.1. With the previous notations, the aggregation of the family $(R_i)_{i \in [a, b]}$ with respect to T is the fuzzy relation R defined for all $x, y \in X$ by

$$R(x, y) = t^{[-1]} \left(\int_a^b t(R_i(x, y)) di \right)$$

This definition is independent of the generator of T , thanks to the linearity of integration. It is the continuous version of Proposition 3.2. and it allow us to aggregate continuous families of T -transitive fuzzy relations using the given t-norm T . With more reason than in the finite case, we will obtain very low values for $R(x, y)$ in general and if the t-norm is non strict, then it is most likely that most of them be zero. Therefore, we need to find an aggregation generalizing the mean of the finite case:

Definition 4.2. With the previous notations, the mean aggregation of the family $(R_i)_{i \in [a, b]}$ with respect to T is the fuzzy relation R defined for all $x, y \in X$ by

$$R(x, y) = t^{[-1]} \left(\frac{1}{b-a} \int_a^b t(R_i(x, y)) di \right)$$

This definition is also independent of the generator of T , thanks to the linearity of integration.

Probably the most important necessity of aggregating a non-finite family of fuzzy relations is when we need to compare two fuzzy subsets μ and ν of our universe X (i.e.: calculating their degree of similarity or equality or calculating the degree in which one is contained in the other.)

Again, one of the most popular ways is to compare $\mu(x)$ and $\nu(x)$ for all $x \in X$ using \vec{T} or \hat{T} and then taking the infimum of all the results.

Definition 4.3. Let μ, ν be two fuzzy subsets of a set X and T a t-norm. The degree of similarity $E_T(\mu, \nu)$ between μ and ν is defined by

$$E_T(\mu, \nu) = \inf_{x \in X} \vec{T}(\mu(x), \nu(x)).$$

Definition 4.4. Let μ, ν be two fuzzy subsets of a set X . The degree of inclusion $P_T(\mu, \nu)$ of μ into ν is defined by

$$P_T(\mu, \nu) = \inf_{x \in X} \hat{T}(\mu(x) | \nu(x)).$$

It is worth noticing that E_T and P_T are a T -indistinguishability operator and a fuzzy T -preorder on the set $X^{[0,1]}$ respectively.

Both definitions suffer from the drastic effect of the infimum as it is shown in the following example.

Example 4.5. If we have two fuzzy subsets μ, ν of a set X with $\mu(x) = \nu(x)$ for all $x \in X$ except for a value x_0 for which $\mu(x_0) = 1$ and $\nu(x_0) = 0$, then $E_T(\mu, \nu) = 0$ and $P_T(\mu, \nu) = 0$ which means that both subsets are considered completely different or dissimilar and that μ can not be considered as contained in ν at any degree.

Therefore, an average of the values obtained for every $x \in X$ seems a suitable alternative.

Definition 4.6. Let μ, ν be two integrable fuzzy subsets of an interval $[a, b]$ of the real line. The

averaging degree of similarity $E_T^A(\mu, \nu)$ between μ and ν is defined by

$$E_T^A(\mu, \nu) = t^{[-1]} \left(\frac{1}{b-a} \int_a^b t(E_T^x(\mu, \nu)) dx \right).$$

or equivalently

$$E_T^A(\mu, \nu) = t^{[-1]} \left(\frac{1}{b-a} \int_a^b t(\hat{T}(\mu(x), \nu(x))) dx \right).$$

Proposition 4.7. The fuzzy relation E_T^A defined on the set of integrable fuzzy subsets $\mathcal{FI}_{[a,b]}$ of an interval $[a, b]$ of the real line is a T -indistinguishability operator and does not depend on the choice of the additive generator t of the t -norm.

Definition 4.8. Let μ, ν be two integrable fuzzy subsets of an interval $[a, b]$ of the real line. The averaging degree of inclusion $P_T^A(\mu, \nu)$ of μ into ν is defined by

$$P_T^A(\mu, \nu) = t^{[-1]} \left(\frac{1}{b-a} \int_a^b t(P_T^x(\mu, \nu)) dx \right).$$

or equivalently

$$P_T^A(\mu, \nu) = t^{[-1]} \left(\frac{1}{b-a} \int_a^b t(\hat{T}(\mu(x)|\nu(x))) dx \right).$$

Proposition 4.9. The fuzzy relation P_T^A defined on the set of integrable fuzzy subsets of an interval $[a, b]$ of the real line is a fuzzy T -preorder and does not depend on the choice of the additive generator t of the t -norm.

With these definitions both, the degree of similarity and the degree of inclusion of the example 4.5 are 1, which is a very intuitive result.

Example 4.10. Let us consider the two fuzzy subsets of the interval $[0, 2]$ $\mu(x) = 1/2$ and $\nu(x) = x/2 \forall x \in [0, 2]$. Let T_α be the Yager family of t -norms ($T_\alpha(x, y) = 1 - \text{Min}(1, ((1 - x)^\alpha + (1 - y)^\alpha))^{\frac{1}{\alpha}}$) with $\alpha \in (0, \infty)$.

$$P_{T_\alpha}^A(\mu, \nu) = 1 - \left(-\frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} - \left(\frac{1}{2}\right)^{\alpha+1} + \frac{1}{\alpha+1} \right)^{\frac{1}{\alpha}}.$$

and

$$E_{T_\alpha}^A(\mu, \nu) = 1 - \left(\frac{1 - \left(\frac{1}{2}\right)^\alpha}{\alpha+1} \right)^{\frac{1}{\alpha}}.$$

If $\alpha = 1$, then T_α is the Lukasiewicz t -norm L and in this case, the previous formulae give

$P_L^A(\mu, \nu) = 7/8$ and $E_L^A(\mu, \nu) = 3/4$ whereas using Definitions 4.3 and 4.4. we obtain $P_L(\mu, \nu) = E_L(\mu, \nu) = 1/2$.

It is interesting to notice that for $\alpha \rightarrow \infty$, the Yager family tends to the Minimum t -norm. If we calculate the limits of $P_L^A(\mu, \nu)$ and $E_L^A(\mu, \nu)$ for $\alpha \rightarrow \infty$ we obtain $\lim_{\alpha \rightarrow \infty} P_L^A(\mu, \nu) = \lim_{\alpha \rightarrow \infty} E_L^A(\mu, \nu) = 0$ which coincide with $P_{Min}(\mu, \nu)$ and $E_{Min}(\mu, \nu)$.

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