Minimisation of the Expected Weighted Number of Jobs Being Late with Fuzzy Processing Time in a One Machine System

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Abstract

One machine scheduling problem with fuzzy processing times and a penalty for each job being late (the penalty is independent of the magnitude of the lateness) is considered. The optimal sequence is defined as one minimizing the expected value of the total penalty (the expected weighted number of jobs being late). Two special cases are discussed in which the problem can be reduced to well-known and easy to solve optimization problems.

Keywords: scheduling, fuzzy processing time

1 Introduction

We consider a one-machine scheduling problem with fuzzy processing times. This problem has already been studied in the literature [2],[3], but optimal processing sequence can be determined using various criteria. Here we will be interested in minimising the total penalty paid for jobs being late.

2 Formulation of the problem

Let us consider a one machine system in which n jobs $Z_1, Z_2, \ldots, Z_n$ should be processed.

For each job Z_s the following information is given:

- $\tilde{T}_s$ - fuzzy normalized job processing time being a fuzzy number of the L-R type with the membership function $\mu_{\tilde{T}_s}(x)$
- $\tilde{D}_s$ - fuzzy normalized job due date being a fuzzy number of the L-R type with the membership function $\mu_{\tilde{D}_s}(x)$, including the special (crisp) case when $\mu_{\tilde{D}_s}(x)$ is equal to 1 for exactly one x and to 0 elsewhere;
- w_s - penalty which is to be paid when the job is late (independently of the value of the tardiness).

We assume that all the jobs are ready to be processed in moment 0.

The aim is to determine such a processing schedule $(Z_i, Z_{i_2}, \ldots, Z_{i_n})$ for which the expected value of the total penalty, being a fuzzy set, will be minimal (expected value for fuzzy sets is defined in the literature [1]).

3 Weighted number of late jobs

Let us denote with $\tilde{Y}_{Z_s(k)}(p)$ the moment when, for a given permutation $p = (Z_{i_1}, Z_{i_2}, \ldots, Z_{i_{k-1}}, Z_s, Z_{i_{k+1}}, \ldots, Z_{i,_n})$, the processing time of job Z_s on the k-th position is finished.

$$\tilde{Y}_{Z_s(k)}(p) = \tilde{T}_{i_1} \tilde{+} \tilde{T}_{i_2} \tilde{+} \ldots \tilde{+} \tilde{T}_{i_{k-1}} \tilde{+} \tilde{T}_s .$$

$\tilde{Y}_{Z_s(k)}(p)$ is a fuzzy number of the L-R type with the membership function $\mu_{\tilde{Y}_{Z_s(k)(p)}}(x)$.

Let us define function

$$L(x,y)=\begin{cases}1 & for\ x>y\\ 0 & for\ x\le y\end{cases} \qquad (1)$$

If we consider, for a fixed k, all $x\in \tilde{Y}_{Z_s(k)}(p)$ and all $y\in \tilde{D}_s$, function (1) generates for a given permutation p, according to the Zadeh extension principle, fuzzy set of the following form:

$$\tilde{L}_{Z_s(k)}(p)=0/\mu_{0_{Z_s(k)}}(p)+1/\mu_{1_{Z_s(k)}}(p) \quad (2)$$

where:

$$\mu_{0_{Z_s(k)}}(p)=\sup_{x\le y}\min\left(\mu_{\tilde{Y}_{Z_s(k)}(p)}(x),\mu_{\tilde{D}_s}(y)\right)$$

$$\mu_{1_{Z_s(k)}}(p)=\sup_{x>y}\min\left(\mu_{\tilde{Y}_{Z_s(k)}(p)}(x),\mu_{\tilde{D}_s}(y)\right)$$

The fuzzy set (2) takes on value 1 if job Z_s is late and value 0 if it is not late. It is easy to calculate a fuzzy set form of the number of late jobs for a fixed permutation.

Let us denote with $\tilde{K}(p)$ the fuzzy set representing the total penalty we will pay for the jobs being late for permutation p:

$$\tilde{K}(p)=\sum_{k=1}^{n}\sum_{s=1}^{n} w_s\cdot \tilde{L}_{Z_s(k)}(p)\cdot x_{sk}(p)$$

where:

$$\sum_{s=1}^{n} x_{sk}(p)=1 \quad \text{and} \quad \sum_{k=1}^{n} x_{sk}(p)=1$$

and $x_{sk}(p)$ takes on value 1 when job Z_s is processing on the k-th position for permutation p and value 0 otherwise.

As mentioned above, our optimality criterion is the expected value of $\tilde{K}(p)$, which takes on the following form:

$$E\left(\tilde{K}(p)\right)=\sum_{k=1}^{n}\sum_{s=1}^{n} w_s\cdot E\left(\tilde{L}_{Z_s(k)}(p)\right)\cdot x_{sk}(p)$$

Since

$$E\left(\tilde{L}_{Z_s(k)}(p)\right)=\frac{1}{2}\left(1+\mu_{1_{Z_s(k)}}(p)-\mu_{0_{Z_s(k)}}(p)\right)$$

$$for\ s=1,2,\dots,n. \qquad (3)$$

we have

$$E\left(\tilde{K}(p)\right)=\frac{1}{2}\cdot\sum_{s=1}^{n} w_s\cdot x_{ks}+$$

$$\frac{1}{2}\cdot\sum_{k=1}^{n}\sum_{s=1}^{n}\left[w_s\cdot\left(\mu_{1_{Z_s(k)}}(p)-\mu_{0_{Z_s(k)}}(p)\right)\cdot x_{ks}\right].$$

4 Determination of the optimal processing schedule

The optimal processing sequence p^* is such a sequence for which the following condition is fulfilled:

$$E\left(\tilde{K}(p^*)\right)=\min_p\left[E\left(\tilde{K}(p)\right)\right]$$

Let us consider two permutations:

$$p'=\left(Z_{i_1},Z_{i_2},\dots,Z_{i_{k-1}},Z_i,Z_j,Z_{i_{k+2}},\dots,Z_{i_n}\right),$$

$$p''=\left(Z_{i_1},Z_{i_2},\dots,Z_{i_{k-1}},Z_j,Z_i,Z_{i_{k+2}},\dots,Z_{i_n}\right).$$

The only difference between the two sequences are the positions of Z_i and Z_j on the k-th and $k+1$ positions.

Theorem 1: Job Z_i should be processed before job Z_j if

$$\frac{1}{2}\cdot w_i\cdot\left(1+\mu_{1_{Z_i(k+1)}}(p'')-\mu_{0_{Z_i(k+1)}}(p'')\right)+$$

$$\frac{1}{2}\cdot w_j\cdot\left(1+\mu_{1_{Z_j(k)}}(p'')-\mu_{0_{Z_j(k)}}(p'')\right)-$$

$$\frac{1}{2}\cdot w_i\cdot\left(1+\mu_{1_{Z_i(k)}}(p')-\mu_{0_{Z_i(k)}}(p')\right)-$$

$$\frac{1}{2}\cdot w_j \cdot \left(1+\mu_{1_{Z_j(k+1)}}(p') - \mu_{0_{Z_j(k+1)}}(p')\right) \geq 0$$

otherwise job Z_j should be processed before job Z_i .

Proof: Sequence p' is better than p'' if

$$E\left(\tilde{K}(p'')\right) - E\left(\tilde{K}(p')\right) \geq 0$$

The proof follows immediately from (3) and the formula which comes immediately after it.

In the following we consider two special cases, which may occur pretty often in practice. For those special cases the optimal sequence can be found by well-known, polynomial algorithms.

In the first case we assume that all the jobs Z_s, $s = 1,2,\ldots,n$, have the same processing times $\tilde{T}(x)$. If it is so, it is easy to notice that $\tilde{L}_{Z_s(k)}(p)$ is the same for all the permutations in which job Z_s is served on the k-th position. In this case we can introduce the notation $\tilde{L}_{Z_s(k)}$, denoting the common value of $\tilde{L}_{Z_s(k)}(p)$ for all the permutations:

$$\left(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_{k-1}}, Z_s, Z_{i_{k+2}}, \ldots, Z_{i_n}\right)$$

The following is true:

$$\tilde{L}_{Z_s(k)} = 0/\mu_{0_{Z_s(k)}} + 1/\mu_{1_{Z_s(k)}}$$

Proposition 1: If all the jobs Z_s, $s = 1,2,\ldots,n$.., have the same processing times $\tilde{T}(x)$, then the optimal sequence according to the total penalty expected value criterion can be found by means of a minimal assignment algorithm (e.g. the Hungarian method), where job $Z_1, Z_2, \ldots, Z_n$ are assigned to their position in the optimal sequence and the elements c_{sk} of the assignment matrix will be equal to

$$E\left(w_s \cdot \tilde{L}_{Z_s(k)}\right) = w_s \cdot E\left(\tilde{L}_{Z_s(k)}\right).$$

Proof: For each permutation we have:

$$\min_p E\left(\tilde{K}(p)\right) =$$

$$\min_p \sum_{k=1}^{n}\sum_{s=1}^{n} w_s \cdot E\left(\tilde{L}_{Z_s(k)}\right)\cdot x_{sk}(p) =$$

$$\min_{x_{sk}} \sum_{k=1}^{n}\sum_{s=1}^{n} \left[w_s \cdot E\left(\tilde{L}_{Z_s(k)}\right)\right]\cdot x_{sk}$$

where:

$$\sum_{s=1}^{n} x_{sk} = 1 \quad \text{and} \quad \sum_{k=1}^{n} x_{sk} = 1$$

which completes the proof.

Example

Let us consider jobs Z_1, Z_2, Z_3. For each job Z_s its processing time $\tilde{T}_s$ is a triangular fuzzy number $\mu_{\tilde{T}_s}(x)$=(1,3,2). The due dates are crisp and take on, respectively, the following values $\tilde{D}_1 = (4,4,4)$, $\tilde{D}_2 = (6,6,6)$, $\tilde{D}_3 = (8,8,8)$. The penalties are as follows: $w_1 = 200$, $w_2 = 500$, $w_3 = 300$. All the jobs are ready to be processed in the moment 0.

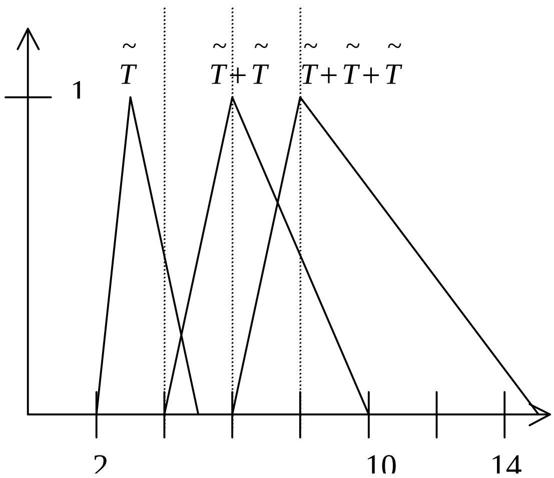

Fig.1: Illustration of the example

The expected values of the penalties, according the job position, are presented in the following table (they result from formula (3)):

$Z_s \backslash$ k	1	2	3
Z_1	$\frac{1}{4}w_1$	w_1	w_1
Z_2	0	$\frac{1}{2}w_2$	w_2
Z_3	0	$\frac{1}{4}w_3$	$\frac{2}{3}w_3$

Substituting the penalty values from the example, we obtain the following assignment matrix:

$Z_s \backslash$ k	1	2	3
Z_1	150	200	200
Z_2	0	250	500
Z_3	0	75	200

The sequence minimizing the expected penalty value is the following one: $p^* = (Z_2, Z_3, Z_1)$. The optimal expected penalty value is equal to 275.

In the other special case the processing times are also equal for all the jobs, but additionally all the jobs have the same due dates. The following proposition is true:

Theorem 2: If all the jobs have the same processing times and the same due dates, then the optimal sequence according to the total penalty expected value criterion can be found by scheduling the jobs in the order of non-increasing values of penalties w_s.

Proof follows immediately from Theorem 1, taking into account that in expressions $\mu 0_{Z_S(k)}(p)$ and $\mu 1_{Z_S(k)}(p)$ the dependence on permutation *p* and number of job *s* disappears.

Conclusions

We have considered one machine scheduling problem with fuzzy processing times and a penalty for each job being late (the penalty is independent of the magnitude of the lateness). We have shown how to determine the optimal sequence according to the criterion of the expected value of the total penalty. For two special cases we have shown how the optimal schedule can be determined effectively by means of well known algorithms. Further research is needed to study other cases.
If the penalties are equal, then the optimal sequence is one minimising the expected number of late jobs. Apart from the expected number of late jobs, it would be interesting to study the fuzzy set form of the number of late jobs. In a following paper we will present a simple algorithm determining this set.

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