Transitivity of Fuzzy Strict Preference Relations in Absence of Incomparability

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Abstract

The aim of this work is to study an essential property in the framework of fuzzy preference structures, which is the transitivity. In particular, we discuss the relationship between the transitivity of a fuzzy large preference relation R and the transitivity of the fuzzy strict preference relation P obtained from R when every pair of elements can be compared. We consider some of the most important types of transitivity for the fuzzy large preference relation R and identify the strongest type of transitivity of the corresponding fuzzy strict preference relation P.

Keywords: Transitivity, fuzzy preference structure, t-norm.

1 Introduction

Transitivity is an essential property in preference modelling and therefore this concept plays an important role in the study of preference structures. These structures are very interesting in decision making theory since they contribute to establish some kind of order in those sets where they are defined.

In the classical theory, when the relations considered are crisp, the transitivity of a large preference relation R can be characterized by the transitivity of the corresponding indifference relation I and strict preference relation P when the relation R is complete [9].

In previous works [5, 6, 7] we have studied the implication between the transitivity of R and the transitivity of I. Now, we consider preference structures in the context of fuzzy relations and we focus on the propagation of the T-transitivity of a fuzzy large preference relation R to the corresponding fuzzy preference relation P, constructed from R. In recent years, some authors as Dasgupta and Deb [3, 4] or Van de Walle [10] have studied this implication considering different situations. In their works they have observed when P satisfies some fixed types of transitivity if R (the relation from which it is defined) satisfies some special definitions of transitivity. In this work we not only investigate whether P is also T-transitive, but we obtain the strongest type of transitivity P can exhibit, from different types of transitivity of R described by some of the most important t-norms.

The work is divided up into 5 sections. In Section 2 we briefly present the classical theory and the equivalence obtained in that case. In Section 3 we introduce the fuzzy theory and some results concerning the concept of completeness. Section 4 includes our results concerning the transitivity of P. In Section 5 we comment some interesting properties of the operators obtained in Section 4.

2 Classical theory

A large preference relation R defined on a set of alternatives A is just a *reflexive* relation interpreted as follows:

$$aRb \text{ if and only if } a \text{ is at least as good as } b\,.$$

Any relation, R, in the previous conditions can be decomposed into disjoint parts: an irreflexive and asymmetric strict preference component P, a reflexive and symmetric indifference component I and an irreflexive and symmetric incomparability component J such that those three relations and the transpose of P, denoted P^t, are a partition of A^2 ($P \cup P^t \cup I \cup J = A^2$). These components can be obtained by considering various intersections: $P = R \cap R^d$, $I = R \cap R^t$ and $J = R^c \cap R^d$ (where ^{c} denotes the complement of a relation and ^{d} denotes the dual of a relation, i.e. the complement of its transpose). Once we have those three components, the relation R can be rebuilt from P and I as their union, $R = P \cup I$.

If we denote the transitivity of a binary relation Q as $Q \circ Q \subseteq Q$, the characterization of the transitivity of a large preference relation R can be written as follows:

Theorem 2.1 [9] *For any reflexive binary relation R with corresponding preference structure (P, I, J) it holds that*

$$R \circ R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ P \subseteq P \\ I \circ I \subseteq I \\ P \circ I \subseteq P \\ I \circ P \subseteq P \end{cases} .$$

When R is complete (i.e. aRb or bRa for any $a, b \in A$) the previous characterization can be simplified. It is important to note that the completeness of R is equivalent to the emptiness of the incomparability relation ($J = \emptyset$). When this holds, the following characterization can be proved

Theorem 2.2 [9] *For any complete binary relation R with corresponding preference structure $(P, I, \emptyset)$ it holds that*

$$R \circ R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ P \subseteq P, \\ I \circ I \subseteq I. \end{cases}$$

3 Fuzzy Preference Structures

In fuzzy preference modelling, a reflexive binary fuzzy relation R on A can also be decomposed into what is called an additive fuzzy preference structure, by means of an (indifference) generator i, which was defined in [1] as follows

Definition 3.1 *A* generator i *is a symmetric (commutative)* $[0,1]^2 \to [0,1]$ *mapping bounded by the Łukasiewicz t-norm, $T_{\mathbf{L}}$, and the minimum operator, $T_{\mathbf{M}}$, i.e. $T_{\mathbf{L}} \leq i \leq T_{\mathbf{M}}$.*

Given a large preference relation R and a generator i, the three components of an additive fuzzy preference structure are defined as follows:

$$\begin{aligned} P(a,b) &= R(a,b) - i(R(a,b), R(b,a)) \\ I(a,b) &= i(R(a,b), R(b,a)) \\ J(a,b) &= I(a,b) - (R(a,b) + R(b,a) - 1). \end{aligned}$$

An additive fuzzy preference structure (AFPS) on A, (P, I, J), is characterized as a triplet of binary fuzzy relations on A such that I is reflexive and symmetric and

$$P(a,b) + P(b,a) + I(a,b) + J(a,b) = 1\,,$$

for every pair of alternatives a and b in A. The corresponding fuzzy large preference relation R is then given by $R(a,b) = P(a,b) + I(a,b)$.

The most popular type of transitivity of fuzzy relations is T-transitivity, with T a t-norm. A binary fuzzy relation Q on A is called T-transitive if it holds that

$$T(Q(a,b), Q(b,c)) \leq Q(a,c)$$

for any $a, b, c \in A$.

T-transitivity can be expressed equivalently as a relational inequality: $Q \circ_T Q \subseteq Q$.

Although traditionally the transitivity of a fuzzy relation is only defined for t-norms, the concept can be extended to any binary operator f defined from [0,1] to [0,1] just by changing T by f: a binary fuzzy relation Q is f-transitive if and only if

$$f(Q(a,b), Q(b,c)) \leq Q(a,c)$$

for any $a, b, c \in A$.

As far as we know, the only generalization of Theorems 2.1 and 2.2 has been obtained in the case of a strongly complete fuzzy large preference relation R (i.e. $\max(R(a,b), R(b,a)) = 1$ for any $a, b \in A$). Note that in that case any generator i leads to the same AFPS and that again $J = \emptyset$.

Theorem 3.1 [2] *Let R be a strongly complete binary fuzzy relation with corresponding fuzzy preference structure $(P, I, \emptyset)$. For any t-norm $T \geq T_{\mathbf{L}}$ it holds that:*

$$R \circ_T R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_T I \subseteq I \\ P \circ_{T_{\mathrm{L}}} I \subseteq P \\ I \circ_{T_{\mathrm{L}}} P \subseteq P \end{cases}.$$

In this work we have focused our study on a more general type of completeness. We have considered fuzzy preference structures without incomparability, this is, structures which relation J is empty. The following Theorem shows that these structures can be characterized by means of a condition on the relation R from which the structure is obtained and a restriction on the generator i.

Theorem 3.2 *Let R be a binary reflexive relation and J the incomparability relation associated to R by a generator i. The following equivalence holds:*

$$J = \emptyset \Longleftrightarrow \begin{cases} R \text{ is weakly complete} \\ i = T_L \end{cases}$$

where a relation R defined on A is weakly complete if $R(a,b) + R(b,a) \geq 1$, $\forall (a,b) \in A^2$.

4 On the transitivity of P

In this section we consider four very important t-norms included in the literature (see, for instance, [8]) to define the transitivity of the preference relation R and we obtain the strongest types of transitivity that P satisfies in those four cases. These t-norms are the Lukasiewicz t-norm (T_{L}), the product t-norm (T_{P}), the minimum t-norm (T_{M}) and the minimum nilpotent t-norm (T_{nM}). The first three ones are important not only because of their good behaviour with respect to many properties but they also allows to build every continuous t-norm. The last one is maybe the most important example of non-continuous t-norm.

Since the family of t-norms is not always wide enough to describe some kinds of transitivity, as we showed in [6], we have not restricted the study to the context of t-norms but we have considered general binary operators on $[0,1]$ to describe the transitivity of P. Since the aim of the work was to obtain the greatest type of transitivity of P, we have not even imposed 1 as neutral element since this restriction implies an upper bound to the transitivity of P: the T_{M}-transitivity. This general framework has allowed to obtain some special functions as we show below.

We begin our study by the weakest one of the four t-norms included in the work: T_{L}.

Proposition 4.1 *The T_{L}-transitivity of P is the strongest type of transitivity that can be obtained from the T_{L}-transitivity of the reflexive weakly complete relation R from which P is defined by the generator $i = T_{\mathrm{L}}$.*

It is important to remark that the implication was proved by Van de Walle in his PhD thesis [10]. In this work we have proved that no stronger implication can be obtained.

If we take a look at the implication proved for strongly complete relations R, we can see that for those relations the T_{M}-transitivity of P were assured while now only T_{L}-transitivity can be obtained. This first difference shows the characterization obtained in the strongly complete case for the transitivity of R cannot be immediately translated to this more general framework. In fact, in this more general context, even when we consider the strongest t-norm to describe the transitivity of R, the strongest function obtained describing the transitivity of P is 0 below the diagonal; it verifies $f(x,y) = 0$ whenever $x + y \leq 1$. This is a much weaker implication than the obtained in the strongly complete case.

We will consider now the t-norm T_{P}.

Proposition 4.2 *Let R be a reflexive weakly complete binary fuzzy relation and let P be the strict preference relation of the FPS obtained from R by the generator T_{L}. Then*

$$R \circ_{T_{\mathrm{P}}} R \subseteq R \quad \Rightarrow \quad P \circ_{f_{\mathrm{PL}}} P \subseteq P,$$

where

$$f_{\mathrm{PL}}(x,y) = \begin{cases} 0 & \text{if} \quad \min\{x,y\} = 0\,, \\ \frac{\max\{x+y-1,0\}}{\min\{x,y\}} & \text{otherwise}\,; \end{cases}$$

and this is the strongest implication that can be proved.

It is interesting to remark that the function f_{PL} obtained above is greater than minimum for some values. This implies that for any t-norm greater than T_{P} describing the transitivity of R the transitivity of P will be greater than minimum at least in some points. This is true for the particular case of the T_{M}-transitivity of R.

Proposition 4.3 *Let R be a reflexive weakly complete binary fuzzy relation and let P be the strict preference relation of the fuzzy preference structure obtained from R by the generator $i = T_{\mathrm{L}}$. Then,*

$$R \circ_{T_{\mathrm{M}}} R \subseteq R \quad \Rightarrow \quad P \circ_{f_{\max}} P \subseteq P$$

and this is the strongest result that can be obtained.

$$f_{\max}(x,y) = \begin{cases} 0 & \text{if} \quad x+y \leq 1\,; \\ \max\{x,y\} & \text{if} \quad x+y > 1\,. \end{cases}$$

Finally we study the result obtained from the T_{nM}-transitive R.

Proposition 4.4 *Let R be a reflexive weakly complete relation and let P be the strict preference relation obtained from R by the generator $i = T_{\mathrm{L}}$. Then*

$$R \circ_{T_{nM}} R \subseteq R \quad \Rightarrow \quad P \circ_{T_{nM}} P \subseteq P\,,$$

and this is the strongest implication that can be proved.

5 Properties of the new operators

The functions T_{L} and T_{nM} obtained in Propositions 4.1 and 4.4 are well known t-norms. Nevertheless f_{PL} and $f_{\max}$ are not t-norms since they are greater (in some points of their domains) than minimum, which is an upper bound of the family of t-norms. Another important observation concerning these two new operators is that they are not continuous despite they are obtained from continuous operators T_{P} and T_{M} respectively. The lack of continuity shows that these two operators do not belong to another important family of operators: they are not copulas.

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