Cardinalities of Fuzzy Sets with Triangular Norms

Maciej Wygralak
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87, 61-614 Poznań, Poland
e-mail: wygralak@math.amu.edu.pl

Abstract

This paper presents a variety of cardinality concepts offered by cardinality theory of triangular norm-based fuzzy sets, including the scalar and the "fuzzy" streams.

Keywords: Cardinality of a fuzzy set, Triangular norm.

1 Introduction

One of the most fundamental mathematical characteristics of a set and, simultaneously, a very important information about it is its cardinality. Exactly the same concerns fuzzy sets. This theoretical motivation for undertaking the question of cardinalities of fuzzy sets is enhanced by multiple applications. Computing with words, communication with data bases and decision making are simple examples of areas in which satisfactory answers to soft cardinal queries are expected (see [2, 12, 13]). We mean queries like "How many x's are $\boldsymbol{p}$?" or "Are there more x's being $\boldsymbol{p}$ than x's being $\boldsymbol{q}$?", i.e. queries about cardinalities of fuzzy sets or comparisons of such cardinalities; $\boldsymbol{p}$, $\boldsymbol{q}$ - generally imprecise properies. Simple instantiations are "How many individuals in the database are rather tall and about 30 years old?" and "In an urn containing balls of various sizes, are there more large balls than small ones?".

Trying to define the notion of a cardinality $|A|$ of a fuzzy set $A\colon \boldsymbol{M} \to [0, 1]$, the main difficulty and difference in comparison with sets is that to belong to A is a matter of degree. Counting and cardinal calculus under fuzziness thus become a task which is more advanced and complicated than in the case of sets. Contemporary fuzzy set cardinality theory offers two main constructive general approaches:

(a) $|A|$ = a *generalized cardinal number*, i.e. a convex fuzzy set of usual cardinal numbers or, in the finite case, a convex fuzzy set in $\mathbb{N} = \{0, 1, 2, \dots\}$,

(b) $|A|$ = a single cardinal number or, in the finite case, a single nonnegative integer or real number.

The "fuzzy" approach in (a), offering the fuzzy perception of cardinality, gives us the most complete and adequate cardinal information about A at the price of relatively high complexity, whereas the approach (b) is scalar, nonfuzzy. The scalar optics of cardinality is simple and convenient in every respect and, therefore, is favoured by many practitioners despite of its disadvantages. Various variants of the approaches (a) and (b) for fuzzy sets with the standard operations are discussed in [1, 2, 8, 12]. In Sections 2 and 3, we like to present a variety of cardinality concepts for fuzzy sets with triangular norms. Our attention will be focused on finite fuzzy sets which play the central role from the viewpoint of applications. From now on, the phrase "fuzzy set" means "finite fuzzy set" if not emphasized otherwise. The family of all (finite) fuzzy sets in the universe $\boldsymbol{M}$ will be denoted by FFS, whereas FCS symbolizes the family of all finite sets in $\boldsymbol{M}$.

Let us recollect those notions and facts concerning triangular norms which will be useful in the main discussion. Further details and references can be found in [5, 6, 11].

One says that a binary operation $\boldsymbol{t}$ in $[0, 1]$ is a *triangular norm* (*t-norm*) if $\boldsymbol{t}$ is commutative, associative, nondecreasing in each argument, and has 1 as neutral element. An analogous operation $\boldsymbol{s}$ having 0 instead of 1 as neutral element is said to be a *triangular conorm* (*t-conorm*). Triangular norms together with triangular conorms will be called *triangular op-*

erations (*t-operations*). If $a\,\boldsymbol{s}\,b = 1-(1-a)\,\boldsymbol{t}\,(1-b)$ for each a and b, we say that $\boldsymbol{s}$ and $\boldsymbol{t}$ are *associated*, and we write $\boldsymbol{s} = \boldsymbol{t}^*$. Simplest instances of a t-norm and the associated t-conorm are $\wedge = \min$ and $\vee = \max$. Also, we shall refer to the following t-norms:

$$a\,\boldsymbol{t}_{\mathrm{a}}\,b = a\,b, \qquad (\textit{algebraic t-norm})$$

$$a\,\boldsymbol{t}_{\text{Ł}}\,b = 0 \vee (a+b-1), \qquad (\textit{Łukasiewicz t-norm})$$

$$a\,\boldsymbol{t}_{\mathrm{Y},p}\,b = 0 \vee [1-((1-a)^p+(1-b)^p)^{1/p}], \quad p>0, \qquad (\textit{Yager t-norms})$$

$$a\,\boldsymbol{t}_{\mathrm{S},p}\,b = [0 \vee (a^p+b^p-1)]^{1/p}, \quad p>0, \qquad (\textit{Schweizer t-norms})$$

$$a\,\boldsymbol{t}_{\mathrm{F},\lambda}\,b = \log_\lambda\Big(1+\frac{(\lambda^a-1)(\lambda^b-1)}{\lambda-1}\Big), \quad 1 \neq \lambda > 0. \qquad (\textit{Frank t-norms})$$

The associated t-conorms will be called and denoted in a similar way replacing the letter $\boldsymbol{t}$ by $\boldsymbol{s}$, e.g. $\boldsymbol{s}_{\text{Ł}} = \boldsymbol{t}_{\text{Ł}}^*$ (*Łukasiewicz t-conorm*) and $\boldsymbol{s}_{\mathrm{S},p} = \boldsymbol{t}_{\mathrm{S},p}^*$ (*Schweizer t-conorm*). Exceptional limit properties of Frank t-operations allow ones to put $\boldsymbol{t}_{\mathrm{F},0} = \wedge$, $\boldsymbol{t}_{\mathrm{F},1} = \boldsymbol{t}_{\mathrm{a}}$, $\boldsymbol{t}_{\mathrm{F},\infty} = \boldsymbol{t}_{\text{Ł}}$, $\boldsymbol{s}_{\mathrm{F},0} = \vee$, $\boldsymbol{s}_{\mathrm{F},1} = \boldsymbol{s}_{\mathrm{a}}$ and $\boldsymbol{s}_{\mathrm{F},\infty} = \boldsymbol{s}_{\text{Ł}}$. The extended families $(\boldsymbol{t}_{\mathrm{F},\lambda})_{\lambda\in[0,\infty]}$ and $(\boldsymbol{s}_{\mathrm{F},\lambda})_{\lambda\in[0,\infty]}$ are called *Frank families* of t-operations.

A continuous t-norm $\boldsymbol{t}$ (t-conorm $\boldsymbol{s}$, resp.) is called *Archimedean* if $a\,\boldsymbol{t}\,a < a$ ($a\,\boldsymbol{s}\,a > a$, resp.) for $a \in (0,1)$. One says that an Archimedean t-operation is *strict* if it is strictly increasing on $(0,1)^2$. Strictly increasing and, in particular, strict t-norms do not have zero divisors ($a, b > 0 \Rightarrow a\,\boldsymbol{t}\,b > 0$). Nonstrict Archimedean t-norms do have, which means that they show some inertia in attaining positive values: a positive argument <1 is still treated as zero whenever the other is not sufficiently "large". $\boldsymbol{t}_{\mathrm{a}}$ and $\boldsymbol{t}_{\text{Ł}}$, resp., are typical examples of a strict t-norm and a nonstrict Archimedean t-norm, resp.

Theorem 1.1 (Ling [7]). (a) *$\boldsymbol{t}$ is an Archimedean t-norm iff there exists a strictly decreasing and continuous $g\colon [0,1] \to [0,\infty]$ such that $g(1) = 0$ and*

$$a\,\boldsymbol{t}\,b = g^{-1}(g(0) \wedge (g(a)+g(b)))$$

for each $a, b \in [0,1]$. $\boldsymbol{t}$ is strict iff $g(0) = \infty$.
(b) *$\boldsymbol{s}$ is an Archimedean t-conorm iff there exists a strictly increasing and continuous $h\colon [0,1] \to [0,\infty]$ such that $h(0) = 0$ and*

$$a\,\boldsymbol{s}\,b = h^{-1}(h(1) \wedge (h(a)+h(b)))$$

for each $a, b \in [0,1]$. $\boldsymbol{s}$ is strict iff $h(1) = \infty$.

Generators of Archimedean t-operations, the functions g and h from Theorem 1.1, are uniquely determined up to a positive constant factor. So, considering nonstrict Archimedean t-operations, we can restrict ourselves to their *normed* generators, i.e. to g and h such that $g(0) = 1$ and $h(1) = 1$.

Theorem 1.2 (Frank [4]). *A continuous t-norm $\boldsymbol{t}$ and a continuous t-conorm $\boldsymbol{s}$ fulfil the functional equation*

$$\forall a, b \in [0,1]\colon\ a\,\boldsymbol{t}\,b + a\,\boldsymbol{s}\,b = a+b$$

iff $\boldsymbol{t} = \boldsymbol{t}_{\mathrm{F},\lambda}$ and $\boldsymbol{s} = \boldsymbol{s}_{\mathrm{F},\lambda}$ for some $\lambda \in [0,\infty]$ or $\boldsymbol{t}$ is the ordinal sum of a family $((\boldsymbol{t}_{\mathrm{F},\lambda}, [a_\lambda, b_\lambda]))_{\lambda\in \boldsymbol{J}\subset(0,\infty]}$, whereas $\boldsymbol{s}$ is determined via the above equation.

Each nonincreasing function $\nu\colon [0,1] \to [0,1]$ with $\nu(0) = 1$ and $\nu(1) = 0$ will be called a *negation*. The negation ν_* with $\nu_*(a) = 0$ for $a > 0$ is thus the smallest possible negation, whereas the largest possible one is ν^* such that $\nu^*(a) = 1$ for $a < 1$. Strictly decreasing and continuous negations are called *strict negations*. One says that a negation ν is strong if it is strict and involutive ($\nu(\nu(a)) = a$). A typical example of a strong negation is the *Łukasiewicz negation* $\nu_{\text{Ł}}$ with $\nu_{\text{Ł}}(a) = 1-a$. Let

$$\nu_{\boldsymbol{t}}(a) = \bigvee\{c \in [0,1]\colon a\,\boldsymbol{t}\,c = 0\}$$

and

$$\nu_{\boldsymbol{s}}(a) = \bigwedge\{c \in [0,1]\colon a\,\boldsymbol{s}\,c = 1\}$$

with a t-norm $\boldsymbol{t}$, a t-conorm $\boldsymbol{s}$ and $a \in [0,1]$.

Theorem 1.3. (a) *$\nu_{\boldsymbol{t}}$ and $\nu_{\boldsymbol{s}}$ are negations.*
(b) *If $\boldsymbol{t}$ is strict or $\boldsymbol{t} = \wedge$, then $\nu_{\boldsymbol{t}} = \nu_*$.*
If $\boldsymbol{s}$ is strict or $\boldsymbol{s} = \vee$, then $\nu_{\boldsymbol{s}} = \nu^*$.
(c) *If $\boldsymbol{t}$ is a nonstrict Archimedean t-norm and g is its generator, then the negation $\nu_{\boldsymbol{t}}$ is strong and*

$$\forall a \in [0,1]\colon\ \nu_{\boldsymbol{t}}(a) = g^{-1}(g(0)-g(a)).$$

If $\boldsymbol{s}$ is a nonstrict Archimedean t-conorm with a generator h, then $\nu_{\boldsymbol{s}}$ is strong and

$$\forall a \in [0,1]\colon\ \nu_{\boldsymbol{s}}(a) = h^{-1}(h(1)-h(a)).$$

Strong negations $\nu_{\boldsymbol{t}}$ and $\nu_{\boldsymbol{s}}$ generated by nonstrict Archimedean $\boldsymbol{t}$ and $\boldsymbol{s}$ can be used to establish a correspondence between t-norms and t-conorms from that class via De Morgan-type expressions involving $\nu_{\boldsymbol{t}}$ and $\nu_{\boldsymbol{s}}$:

$$a\,\boldsymbol{t}^\circ\,b = \nu_{\boldsymbol{t}}(\nu_{\boldsymbol{t}}(a)\,\boldsymbol{t}\,\nu_{\boldsymbol{t}}(b)), \quad a\,\boldsymbol{s}^\circ\,b = \nu_{\boldsymbol{s}}(\nu_{\boldsymbol{s}}(a)\,\boldsymbol{s}\,\nu_{\boldsymbol{s}}(b)).$$

The binary operations $\boldsymbol{t}^\circ$ and $\boldsymbol{s}^\circ$, resp., are then a nonstrict Archimedean t-conorm and a nonstrict Archimedean t-norm, resp. One says that $\boldsymbol{t}$ and $\boldsymbol{s}$ are *complementary* if $\boldsymbol{s} = \boldsymbol{t}^\circ$. Instances of pairs of such operations are $(\boldsymbol{t}_{\mathrm{S},p}, \boldsymbol{s}_{\mathrm{Y},p})$ and $(\boldsymbol{t}_{\mathrm{Y},p}, \boldsymbol{s}_{\mathrm{S},p})$.

Basic binary operations on (arbitrary) fuzzy sets with t-norms and t-conorms will be denoted by $\cup_{\boldsymbol{s}}$, $\cap_{\boldsymbol{t}}$ and $\times_{\boldsymbol{t}}$. A^ν symbolizes the complement of A induced by a negation ν; $A^\nu(x) = \nu(A(x))$. $\cup = \cup_\vee$, $\cap = \cap_\wedge$, $\times = \times_\wedge$, and $' = {}^\nu$ with $\nu = \nu_{Ł}$ are the standard operations. Let $[A]_k = \bigvee\{t\colon |A_t| \geq k\}$ with $k \in \mathbb{N}$ and $A_t = \{x\colon A(x) \geq t\}$ (*t-cut set*); $A^t = \{x\colon A(x) > t\}$ (*sharp t-cut set*). Throughout, $n = |\operatorname{supp}(A)|$ and $m = |\operatorname{core}(A)|$. Since we assume that A is finite, $[A]_k$ with $0 < k \leq n$ is the kth element in the nonincreasingly ordered sequence of all positive values $A(x)$, including their possible repetitions.

2 Triangular norm-based generalized cardinals

Generalized cardinals will be denoted by lowercase letters $\alpha, \beta, \ldots$ from the beginning of the Greek alphabet. The equality $\alpha = \beta$ of two generalized cardinals is understood in the usual pointwise way. If α expresses the cardinality of A, we write $\alpha = |A|$. Let us present three main classes of generalized cardinals of fuzzy sets with triangular norms ([10, 11]).

2.1 Generalized FGCounts

Let

$$\alpha(k) = [A]_1\, \boldsymbol{t}\, [A]_2\, \boldsymbol{t} \ldots \boldsymbol{t}\, [A]_k$$

with a t-norm $\boldsymbol{t}$ and $k \in \mathbb{N}$. α is then called the *generalized FGCount of A* (more precisely, the generalized FGCount of A generated by $\boldsymbol{t}$). In the language of many-valued logic, $\alpha(k)$ expresses a degree to which A contains (at least) i elements for each $i \leq k$ with the quantification "for each" interpreted via $\boldsymbol{t}$. If $\boldsymbol{t} = \wedge$, α becomes the usual FGCount of A from [12]. Put $\boldsymbol{k}_* = 1_{\{0, 1, \ldots, k\}}$ with $k \in \mathbb{N}$.

Theorem 2.1. *Let* $A \in \mathrm{FFS}$ *and* $|A| = \alpha$ *with an arbitrary t-norm* $\boldsymbol{t}$. *We then have*:
(a) α *is normal and convex.*
(b) $\alpha(k) = 1$ *for* $k \leq m$, $\alpha(k) = 0$ *for* $k > n$.
(c) $\alpha = \boldsymbol{n}_*$ *iff* $A \in \mathrm{FCS}$.

Our further discussion will be restricted to the class $Atn^\wedge$ of Archimedean t-norms including the t-norm $\wedge$. $Stn^\wedge$ will denote its subclass composed of strict t-norms and $\wedge$, whereas *Natn* symbolizes the family of nonstrict Archimedean t-norms.

Let $\boldsymbol{t} \in Atn^\wedge$, $\alpha = |A|$ and $\beta = |B|$ for $A, B \in \mathrm{FFS}$. The equipotency relation ~ guaranteeing that $\alpha = \beta$ iff $A \sim B$ is of the form

$$A \sim B \;\Leftrightarrow\; e(A) = e(B) \ \&\ \forall k \leq e(A)\colon [A]_k = [B]_k$$

with $e(A) = \bigvee\{k \in \mathbb{N}\colon [A]_1\, \boldsymbol{t}\, [A]_2\, \boldsymbol{t} \ldots \boldsymbol{t}\, [A]_k > 0\}$. The equipotency $A \sim B$ thus means that "large" membership values in A are identical to "large" membership values in B. In other words, t-cut sets (sharp or not) of A with "large" t's are equipotent to corresponding t-cut sets of B. Similarly to the classical theory, the notation $A \sim B$ can be replaced by $|A| = |B|$. If $\boldsymbol{t} \in Stn^\wedge$, ~ becomes less liberal and the dependence on $\boldsymbol{t}$ vanishes, namely we have

$$\begin{aligned} A \sim B &\Leftrightarrow \forall k \in \mathbb{N}\colon [A]_k = [B]_k \\ &\Leftrightarrow \forall t \in (0, 1]\colon |A_t| = |B_t| \\ &\Leftrightarrow \forall t \in [0, 1)\colon |A^t| = |B^t|. \end{aligned}$$

A partial ordering of generalized FGCounts with $\boldsymbol{t} \in Atn^\wedge$ can be defined in the classical-like manner:

$$\begin{aligned} |A| \leq |B| \;&\Leftrightarrow\; \exists B^* \subset B\colon |A| = |B^*|, \\ |A| < |B| \;&\Leftrightarrow\; |A| \leq |B| \ \&\ |A| \neq |B|, \\ \alpha \leq \beta \;&\Leftrightarrow\; \exists A, B\colon |A| = \alpha \ \&\ |B| = \beta \ \& \\ &\qquad\qquad |A| \leq |B|, \\ \alpha < \beta \;&\Leftrightarrow\; \exists A, B\colon |A| = \alpha \ \&\ |B| = \beta \ \& \\ &\qquad\qquad |A| < |B|. \end{aligned}$$

Both $\alpha \leq \beta$ and $\alpha < \beta$ are well-defined inequalities as they do not depend on the choice of $A, B \in \mathrm{FFS}$ such that $|A| = \alpha$ and $|B| = \beta$. If $\boldsymbol{t} = \wedge$, then

$$\alpha \leq \beta \;\Leftrightarrow\; \alpha \subset \beta.$$

For $\boldsymbol{t} \neq \wedge$,

$$\alpha \leq \beta \;\Rightarrow\; \alpha \subset \beta.$$

The operations of addition and multiplication of generalized FGCounts α and β with $\boldsymbol{t} \in Atn^\wedge$ are also defined in the classical-like way: if $\alpha = |A|$ and $\beta = |B|$ with $A, B \in \mathrm{FFS}$, we put

$$\begin{gathered} \alpha + \beta = |A \cup B| \;\text{ provided that }\; A \cap B = 1_\varnothing, \\ \alpha\beta = |A \times B|. \end{gathered}$$

Neutral elements of those operations are $\mathbf{0}_*$ and $\mathbf{1}_*$, respectively. Again, the sum $\alpha + \beta$ and the product $\alpha\beta$ are well-constructed as they do not depend on the choice of A and B such that $|A| = \alpha$ and $|B| = \beta$. $\alpha+\beta$ can be equivalently computed via the t-norm-based extension principle, namely

$$(\alpha+\beta)(k) = \bigvee\{\alpha(i)\,\boldsymbol{t}\,\beta(j)\colon i+j=k\},$$

$k\in\mathbb{N}$. In the case of the product $\alpha\beta$, that principle can be used only if $\boldsymbol{t}=\wedge$. The *valuation property*

$$\forall A, B\in \mathrm{FFS}\colon\ |A\cap_{\boldsymbol{t}} B| + |A\cup_{\boldsymbol{s}} B| = |A| + |B|$$

is satisfied by generalized FGCounts with $\boldsymbol{t}\in Atn^{\wedge}$ iff $\boldsymbol{t}=\wedge$ and $\boldsymbol{s}=\vee$.

Theorem 2.2. *Let* α, β, γ *and* δ *denote generalized FGCounts generated by* $\boldsymbol{t}\in Atn^{\wedge}$. *We then have*:
(a) $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$, (*distributivity*)
(b) $(\alpha\le\beta\ \&\ \gamma\le\delta) \Rightarrow (\alpha+\gamma\le\beta+\delta\ \&\ \alpha\gamma\le\beta\delta)$, (*monotonicity*)
(c) $\alpha, \beta\ge\boldsymbol{2}_* \Rightarrow \alpha+\beta\le\alpha\beta$,
(d) $\alpha+\beta=\alpha+\gamma \Rightarrow \beta=\gamma$ *if* $\boldsymbol{t}\in Stn^{\wedge}$,
$\alpha\beta=\alpha\gamma \Rightarrow \beta=\gamma$ *if* $\alpha\ge\boldsymbol{1}_*$ *and* $\boldsymbol{t}\in Stn^{\wedge}$. (*cancellation laws*)

Modifications of the thesis (d) with = replaced by ≤ or < are also valid. Finally, in contrast to the classical-like properties listed in Theorem 2.2, one should emphasize that $\alpha<\gamma \nRightarrow \exists\beta\colon \alpha+\beta=\gamma$. This failure of the *compensation property* does form one of the most important differences between the arithmetic of ordinary cardinals and the arithmetic of generalized FGCounts.

2.2 Generalized FLCounts

Define the generalized cardinal $\alpha = |A|$ of a fuzzy set $A\in\mathrm{FFS}$ in a dual way as

$$\begin{aligned}\alpha(k) &= \nu([A]_{k+1})\,\boldsymbol{t}\,\nu([A]_{k+2})\,\boldsymbol{t}\,\dots\\ &= \nu([A]_{k+1})\,\boldsymbol{t}\,\nu([A]_{k+2})\,\boldsymbol{t}\,\dots\,\boldsymbol{t}\,\nu([A]_{n})\end{aligned}$$

for $k\in\mathbb{N}$ with a triangular norm $\boldsymbol{t}$ and a negation ν. α is now called the *generalized FLCount of* A (generated by $\boldsymbol{t}$ and ν). $\alpha(k)$ is a degree to which A contains at most i elements for each $i\ge k$. Putting $\boldsymbol{t}=\wedge$ and $\nu=\nu_{\text{Ł}}$, α collapses to the classical FLCount of A (see [12]). Let $\boldsymbol{k}^* = 1_{\{k, k+1, \dots\}}$ with $k\in\mathbb{N}$.

Theorem 2.3. *Let* $\boldsymbol{t}$ *and* ν, *resp., denote a t-norm and a negation, resp. The generalized FLCount* α *of* $A\in\mathrm{FFS}$ *fulfils the following properties*:
(a) α *is normal and convex.*
(b) $\alpha(k)=1$ *for* $k\ge n$, $\alpha(k)=0$ *for* $k<m$.
(c) $\alpha=\boldsymbol{n}^*$ *if* $A\in\mathrm{FCS}$.

Restrict our discussion to $\boldsymbol{t}\in Atn^{\wedge}$ and $\nu\in Sng$, where Sng denotes the class of all strict negations. The equipotency relation ~ corresponding to generalized FLCounts is then of the form

$$A\sim B \iff e^*(A)=e^*(B)\ \&\ \forall k>e^*(A)\colon [A]_k=[B]_k$$

with $e^*(A) = \bigwedge\{k\in\mathbb{N}\colon\ \nu([A]_{k+1})\,\boldsymbol{t}\,\nu([A]_{k+2})\,\boldsymbol{t}\,\dots\,\boldsymbol{t}\,\nu([A]_n)>0\}$. If $\boldsymbol{t}\in Natn$, bottom membership values in A and B and their identity are thus essential for having $A\sim B$ (cf. Subsection 2.1). For $\boldsymbol{t}\in Stn^{\wedge}$, the above definition collapses to

$$A\sim B \Leftrightarrow \forall k\in\mathbb{N}\colon [A]_k=[B]_k.$$

Order and basic operations on generalized FLCounts with $\boldsymbol{t}\in Stn^{\wedge}$ are defined as in Subsec. 2.1. Theorem 2.2 with $\boldsymbol{k}_*$ replaced by $\boldsymbol{k}^*$ remains valid for these generalized FLCounts.

2.3 Generalized FECounts

For a t-norm $\boldsymbol{t}$ and a negation ν, the formula

$$\alpha(k) = [A]_1\,\boldsymbol{t}\,\dots\,\boldsymbol{t}\,[A]_k\,\boldsymbol{t}\,\nu([A]_{k+1})\,\boldsymbol{t}\,\dots\,\boldsymbol{t}\,\nu([A]_n)$$

with $k\in\mathbb{N}$ defines the *generalized FECount of* A (generated by $\boldsymbol{t}$ and ν). It is the intersection induced by $\boldsymbol{t}$ of the generalized FGCount and the generalized FLCount of A. In symbols,

$$\mathrm{FECount}_{\boldsymbol{t},\nu}(A) = \mathrm{FGCount}_{\boldsymbol{t}}(A)\cap_{\boldsymbol{t}}\mathrm{FLCount}_{\boldsymbol{t},\nu}(A).$$

If $\boldsymbol{t}=\wedge$ and $\nu=\nu_{\text{Ł}}$, α becomes the usual FECount of A ([12]). $\boldsymbol{t}=\wedge$ and $\nu=\nu^*$ lead to generalized cardinals due to Dubois ([2]). Put $\boldsymbol{k}=1_{\{k\}}$.

Theorem 2.4. *For* $A\in\mathrm{FFS}$, *a t-norm* $\boldsymbol{t}$ *and a negation* ν, *the generalized FECount* $\alpha=|A|$ *satisfies the following*:
(a) α *is convex.*
(b) $\alpha(k)=0$ *for each* $k<m$ *and* $k>n$.
(c) $\alpha=\boldsymbol{n}$ *iff* $A\in\mathrm{FCS}$.
(d) *If* $\nu\in Sng$ *with a fixed point* $a^*\in(0,1)$, *then*

$$\alpha(k)\le\alpha(|A^{a^*}|)=\alpha(|A^{a^*}|+1)=\dots=\alpha(|A_{a^*}|).$$

If $\boldsymbol{t} \in Stn^{\wedge}$ and $\nu \in Sng$, the equipotency relation corresponding to generalized FECounts is again of the form $A \sim B \Leftrightarrow \forall k \in \mathbb{N}\colon [A]_k = [B]_k$. Equipotency of two fuzzy sets thus means equipotency of their corresponding t-cut sets. In particular, equipotent fuzzy sets have equipotent cores as well as equipotent supports. Inequalities between as well as addition and multiplication of generalized FECounts with $\boldsymbol{t} \in Stn^{\wedge}$ and $\nu \in Sng$ are defined as in Subsection 2.1. Properties of these operations, having $\mathbf{0}$ and $\mathbf{1}$ as neutral elements, are identical to those from Theorem 2.2 with $\boldsymbol{k}_*$ replaced by $\boldsymbol{k}$.

The situation is quite different if $\boldsymbol{t} \in Natn$. $\boldsymbol{t}$ then has zero divisors and an equipotency relation similar to that for $\boldsymbol{t} \in Stn^{\wedge}$ cannot be constructed. What is more, we can have $\alpha(k) = 0$ for each k, which means that A is totally dissimilar to any set of any cardinality. Details concerning such singular fuzzy sets are presented in [3].

3 Scalar cardinalities of fuzzy sets with triangular norms

3.1 Cardinal numbers of t-cut sets

One of the simplest concepts of a scalar cardinality of A lies in defining that cardinality as the cardinal number of a t-cut set of A. By Theorem 2.4(d), if ν is a strict negation with a fixed point $a^* \in (0,1)$, the generalized FECount of A generated by a t-norm $\boldsymbol{t}$ and ν attains its maximum on $[|A^{a^*}|, |A_{a^*}|]$. Each cardinal from the interval $[|A^{a^*}|, |A_{a^*}|]$ is thus an equally good candidate for scalar cardinality of A. For $\nu = \nu_{Ł}$, this interval collapses to $[|A^{0.5}|, |A_{0.5}|]$.

3.2 Generalized sigma counts

Let us introduce the following axiomatic definition of scalar cardinalities of fuzzy sets ([9, 11]).

Definition 3.1. A function $\sigma\colon \mathrm{FFS} \to [0, \infty)$ is called a *scalar cardinality* if for each $a, b \in [0,1]$, $x, y \in \boldsymbol{M}$ and $A, B \in \mathrm{FFS}$ the following axioms are satisfied:

(A1) $\sigma(1/x) = 1$,

(A2) $a \le b \Rightarrow \sigma(a/x) \le \sigma(b/y)$,

(A3) $A \cap B = 1_{\varnothing} \Rightarrow \sigma(A \cup B) = \sigma(A) + \sigma(B)$.

One says that $\sigma(A)$ is a *scalar cardinality of A* whenever σ satisfies (A1)-(A3). The following properties are simple consequences of the axioms:

$\sigma(A) = n$ if $A \in \mathrm{FCS}$, (*coincidence*)

$\sigma(A) \le \sigma(B)$ if $A \subset B$, (*monotonicity*)

$m \le \sigma(A) \le n$. (*boundedness*)

Scalar cardinalities from Definition 3.1 do have a convenient characterization described below.

Theorem 3.2. σ *is a scalar cardinality iff there exists a nondecreasing function* $f\colon [0,1] \to [0,1]$ *such that* $f(0) = 0$, $f(1) = 1$ *and*

$$\sigma(A) = \Sigma_{x \in \mathrm{supp}(A)}\, f(A(x)) \quad \textit{for each } A.$$

Consequently, scalar cardinalities from Definition 3.1 can be called *generalized sigma counts* as they are natural generalizations of the sigma count $sc_A = \Sigma_{x \in \mathrm{supp}(A)} A(x)$ of A (see [2, 12]). Each function f satisfying the conditions of Theorem 3.2 is said to be a *cardinality pattern*. It expresses our understanding of the (scalar) cardinality of a singleton. Let us list some basic examples.

(a) $f_{1,t}(a) = (1$ if $a \ge t$, else $0)$ with $t \in (0,1]$. Then

$$\sigma(A) = |A_t|.$$

(b) $f_{2,t}(a) = (1$ if $a > t$, else $0)$ with $t \in [0,1)$. Now

$$\sigma(A) = |A^t|.$$

(c) $f_{3,p}(a) = a^p$ with $p > 0$. Then

$$\sigma(A) = \Sigma_{x \in \mathrm{supp}(A)}(A(x))^p.$$

(d) By definition, normed generators of nonstrict Archimedean t-conorms are cardinality patterns. The usual sigma count is thus a scalar cardinality obtained by applying the normed generator of $\boldsymbol{s}_{Ł}$.

The outlined axiomatic approach brings together all scalar cardinality concepts used in the literature and, moreover, offers infinitely many new options (see [9, 11]). Very different values of $\sigma(A)$ can be generated by various specific cardinality patterns. The choice of a suitable f is problem-dependent.

Theorem 3.3. *The valuation property*

$$\forall A, B \in \mathrm{FFS}\colon \sigma(A \cap_{\boldsymbol{t}} B) + \sigma(A \cup_{\boldsymbol{s}} B) = \sigma(A) + \sigma(B)$$

holds true for a t-norm $\boldsymbol{t}$, *a t-conorm* $\boldsymbol{s}$, *and a cardinality pattern f iff*

$$\forall a, b \in [0,1]\colon f(a\,\boldsymbol{t}\,b) + f(a\,\boldsymbol{s}\,b) = f(a) + f(b).$$

Let us present a few examples of triples $(f, \boldsymbol{t}, \boldsymbol{s})$ satisfying the valuation property:
(a) $(f, \wedge, \vee)$ with any cardinality pattern f,
(b) $(id, \boldsymbol{t}_{\mathrm{F},\lambda}, \boldsymbol{s}_{\mathrm{F},\lambda})$ with $\lambda \in [0, \infty]$ (id - the identity function), which follows from the Frank theorem,
(c) $(h, \boldsymbol{s}^{\circ}, \boldsymbol{s})$, $\boldsymbol{s}$ - a nonstrict Archimedean t-conorm with normed generator h, e.g. $(f_{3,p}, \boldsymbol{t}_{\mathrm{S},p}, \boldsymbol{s}_{\mathrm{Y},p})$.

Theorem 3.4. *The cartesian product rule*

$$\forall A, B \in \mathrm{FFS}\colon\ \sigma(A \times_{\boldsymbol{t}} B) = \sigma(A) \cdot \sigma(B)$$

is satisfied iff f and $\boldsymbol{t}$ *are such that*

$$\forall a, b \in [0, 1]\colon\ f(a\, \boldsymbol{t}\, b) = f(a) \cdot f(b).$$

The following are thus examples of pairs $(f, \boldsymbol{t})$ fulfilling the cartesian product rule:
(a) if $\boldsymbol{t} = \wedge$, then the cartesian product rule holds true iff $f = f_{1,t}$ or $f = f_{2,t}$,
(b) $(e^{-g}, \boldsymbol{t})$ with a strict $\boldsymbol{t}$ generated by g, e.g. $(id, \boldsymbol{t}_{\mathrm{a}})$
(c) if $\boldsymbol{t} = \boldsymbol{t}_{\mathrm{a}}$, the criterion in Th. 3.4 collapses to the Cauchy functional equation. Its unique continuous solutions are the cardinality patterns $f = f_{3,p}$.

Theorem 3.5. *Assume* $\boldsymbol{M}$ *is finite and* ν *denotes a negation. The complementarity rule*

$$\forall A \in \mathrm{FFS}\colon\ \sigma(A) + \sigma(A^{\nu}) = |\boldsymbol{M}|$$

is fulfilled iff f and ν *satisfy the condition*

$$\forall a \in [0, 1]\colon\ f(a) + f(\nu(a)) = 1.$$

So, instances of pairs (f, ν) satisfying the complementarity rule are:
(a) $(f, \nu_{\text{Ł}})$ with any cardinality pattern f having $(0.5, 0.5)$ as a symmetry point of its diagram,
(b) $(h, \nu_{\boldsymbol{s}})$, where $\boldsymbol{s}$ is a nonstrict Archimedean t-conorm with normed generator h.

A unique nontrivial quadruple $(f, \boldsymbol{t}, \boldsymbol{s}, \nu)$ guaranteeing a simultaneous fulfilment of the valuation property and the cartesian product and complementarity rules seems to be $(id, \boldsymbol{t}_{\mathrm{a}}, \boldsymbol{s}_{\mathrm{a}}, \nu_{\text{Ł}})$. $(e^{-g}, \boldsymbol{t}_{\mathrm{F},\lambda}, \boldsymbol{s}_{\mathrm{F},\lambda}, \nu_{\text{Ł}})$ with $\lambda \approx 5.934$ satisfies the cartesian product rule and leads to an approximate fulfilment of the valuation property and the complementarity rule; g denotes the generator of $\boldsymbol{t}_{\mathrm{F},\lambda}$ (see [11]).

Generalized sigma counts are a good basis for defining relative scalar cardinalities in a flexible way as $\sigma(A \mid B) = \sigma(A \cap_{\boldsymbol{t}} B) / \sigma(B)$; $\boldsymbol{t}$ - a t-norm. They can be treated as conditional probabilities.

3.3 Triangular norm-based generalized sigma counts

There exists an obvious connection between the sigma count of A and FGCount(A):

$$sc_A = \Sigma_{k=1}^{n} \mathrm{FGCount}(A)(k).$$

Replacing FGCount(A) by $\mathrm{FGCount}_{\boldsymbol{t}}(A)$, we get the following generalization of sc_A:

$$sc_{A,\boldsymbol{t}} = \Sigma_{k=1}^{n} [A]_1\, \boldsymbol{t}\, [A]_2\, \boldsymbol{t} \dots \boldsymbol{t}\, [A]_k.$$

It leads to two variants of triangular norm-based generalized sigma counts:

$$\sigma(A) = \Sigma_{k=1}^{n} f([A]_1\, \boldsymbol{t}\, [A]_2\, \boldsymbol{t} \dots \boldsymbol{t}\, [A]_k),$$
$$\sigma(A) = \Sigma_{k=1}^{n} f([A]_1)\, \boldsymbol{t}\, f([A]_2)\, \boldsymbol{t} \dots \boldsymbol{t}\, f([A]_k)$$

with a cardinality pattern f. Conversely, the concept of the FGCount can be generalized as follows:

$$\mathrm{FGCount}_f(A)(k) = f([A]_k),$$
$$\mathrm{FGCount}_{f,\boldsymbol{t}}(A)(k) = f([A]_1\, \boldsymbol{t}\, [A]_2\, \boldsymbol{t} \dots \boldsymbol{t}\, [A]_k),$$
$$\mathrm{FGCount}_{\boldsymbol{t},f}(A)(k) = f([A]_1)\, \boldsymbol{t}\, f([A]_2)\, \boldsymbol{t} \dots \boldsymbol{t}\, f([A]_k).$$

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