On modifiers based on n-placed functions

Jorma K. Mattila
Laboratory of Applied Mathematics
Lappeenranta University of Technology
P.O. Box 20, FIN-53851 Lappeenranta
Finland
e-mail: Jorma.Mattila@lut.fi

Abstract

The idea, how to generate modifiers by n-placed functions defined on $[0,1]^n$ are considered. The subject matter of modifiers are fuzzy sets, i.e. membership functions defined on interval $I = [0,1]$. The definition of modifiers is given and the case, how n-placed generator functions fit together with the definition. Some few properties of modifiers are considered. Some examples of generator functions and modifiers generated by them are given. The examples illustrate how graded modifier systems can be created. The place number n of a generator function takes effect to the strength of a modifier.

Keywords: Modifier, n-placed Function, t-norm, t-conorm, DeMorgan Class of Operators.

1 Basic Concepts

The concept of 'modifier' appears in many ways in the scope of fuzzy logic. For example, Prof. L. A. Zadeh used this term already in the early theory of fuzzy logic. The author has studied modifiers and their logics from modal logical point of view and created some logical systems for modifiers basing on relational Kripke structures of graded modalities (see e.g. Mattila [9]). Kortelainen's [3] concept *modified sets* is one example about the use of this term. In the linguistic view, a modifier can be an adjective, or an adverb, or a phase or clause acting as an adjective or adverb. In every case, the basic principle is the same: the modifier adds information to another element in the sentence (Frances Peck, '*Terms of use*', University of Ottawa). Also some fuzzy logic blocs altering the behavior of PID controllers are called modifiers, too.

The author has considered modifiers and modifier logics in several situations (see e.g. J. K. Mattila [6, 7, 8]). Some considerations about modifiers generated by t-norms and t-conorms are done in [8]. After this work Dr. József Dombi suggested the author to study modifiers generated by n-placed functions in the way to use some t-norms and t-conorms generalized for several variables. Some results from these studies are [10] and [11].

The aim is to consider modifiers as operators for modifying fuzzy sets. We refer to Kortelainen's concept *modified sets*. His operators are set functions modifying at first hand ordinary sets. We concentrate here upon modifiers generated by n-placed functions, n = 2, 3, ..., especially using t-norms and t-conorms and their generalizations for more than two-placed cases as basic tools.

We choose the range of fuzzy sets (i.e. membership functions) to be the unit interval $I = [0,1]$, as usually. Thus the set of all fuzzy sets of a non-empty set X is the set I^X (including the usual power-set 2^X (i.e. the set of all characteristic functions of the usual subsets of X) as a special case). It is also a well-known fact that I and I^X are partially ordered sets. In fact, they are also completely distributed complete lattices and Brouwerian lattices (see e.g. Lowen [5]).

These modifiers we consider here are *compositional*, because when we apply a modifier to a fuzzy set we form a composition of two functions.

Definition 1 (Modifier). *We say that a mapping $M : I^X \to I^X$ is (i) a* substantiating modifier *if for any fuzzy set $\mu \in I^X$,*

$$\forall x \in X, M(\mu(x)) \leq \mu(x), \tag{1}$$

(ii) a weakening modifier *if for any $\mu \in I^X$,*

$$\forall x \in X, \mu(x) \leq M(\mu(x)), \tag{2}$$

(iii) an identity modifier if for any $\mu \in I^X$,

$$\forall x \in X, M(\mu(x)) = \mu(x). \tag{3}$$

Identity modifiers are identity mappings on I^X. They are sometimes needed as links between substantiating and weakening modifiers in some logical structures of modifiers.

A given modifier we can associate with the dual modifier according to the following

Definition 2 (Dual Modifier). *Let M and $M*$ be modifiers. We say that $M*$ is the* dual modifier *associated with M, if for any fuzzy set $\mu \in I^X$,*

$$\forall x \in X, M^*(\mu(x)) = n(M(n(\mu(x)))), \tag{4}$$

where n is a strong negation.

Proposition 1 *If M is a substantiating modifier then its dual M^* is a* weakening modifier *and vice versa.*

The proof is given in [10]

The condition

$$\forall x \in X, M^*(\mu(x)) = n(M(n(\mu(x)))) \tag{5}$$

in the previous proof says that the operators M, M^*, and n satisfy DeMorgans law. Thus dual pairs of modifiers with strong negation form classes called DeMorgan classes of operators ([1]). Originally, DeMorgan classes used to consist of a t-norm, corresponding t-conorm, and negation.

We denote α-level set of a fuzzy set μ, as usually,

$$\mu_\alpha = \{x \in X | \mu(x) \geq \alpha, \alpha \in I\}$$

Thus the α-level set of $M(\mu)$ is

$$(M \circ \mu)_\alpha(x) = \{x \in X | M(\mu(x)) \geq \alpha, \alpha \in I\}. \tag{6}$$

It is easy to see that modifiers have following properties. Suppose M is a substantiating modifier. Then we have

$$M(\mathbf{0}_X) = \mathbf{0}_X, \tag{7}$$
$$M^*(\mathbf{1}_X) = \mathbf{1}_X, \tag{8}$$
$$(M^*)^*(\mu(x)) = M(\mu(x)), \tag{9}$$

where $\mathbf{0_X}$ and $\mathbf{1}_X$ are the constant functions $\mathbf{0}_X = 0$ and $\mathbf{1}_X = 1$ for all $x \in X$.

In addition to Def.1, if the kernels of μ and $M \circ \mu$ are the same then we have the following result.

Proposition 2 *Suppose M is a weakening modifier such that $ker(\mu) = ker(M \circ \mu)$, then*

$$M(\mathbf{1}_X) = \mathbf{1}_X. \tag{10}$$

Proof. We have $ker(\mu(x)) = ker(M(\mu(x))) = X$ by the supposition. From this fact the result follows immediately.

In addition to Def.1, if the supports of μ and $M \circ \mu$ are the same then we have the following result.

Proposition 3 *Suppose M is a substantiating modifier such that $supp(\mu) = supp(M \circ \mu)$, then*

$$M(\mathbf{0}_X) = \mathbf{0}_X. \tag{11}$$

Proof. We have $supp(\mu(x)) = supp(M \circ \mu)(x) = \emptyset$ by the supposition. From this fact the result follows immediately.

2 The idea of generalizing modifiers using *n*-ary functions

The simplest idea using n-ary functions for generating modifiers for fuzzy sets is to replace every variable with the membership function of a fuzzy set to be modified. To illustrate the idea, we proceed in the following way. We put the same argument x in every place in the n-tuple of arguments in the function f. Thus we have the generating formulas for substantiating, weakening and identity modifiers, respectively,

$$f(x, x, \ldots, x) \leq x \tag{12}$$

if f generates a substantiating modifier, and

$$f(x, x, \ldots, x) \geq x \tag{13}$$

if f generates a weakening modifier.

An identity operator is generated by any function f, such that

$$f(x,x,\ldots,x) = x, \tag{14}$$

Basing on the formulas (12), (13), and (14) it follows immediately that for all $x \in [0,1]$,

Let $f : [0,1]^n \to [0,1]$ be any n-ary function then f generates a *substantiating modifier* $F(x) = f(x,x,\ldots,x)$ if for all $x_1,x_2,\ldots,x_n \in [0,1]$ the condition

$$f(x_1,x_2,\ldots,x_n) \leq min(x_1,x_2,\ldots,x_n) \tag{15}$$

holds for all $x_i \in [0,1]$. f generates a *weakening modifier* $H(x) = f(x,x,\ldots,x)$ if for all $x_1,x_2,\ldots,x_n \in [0,1]$ the condition

$$f(x_1,x_2,\ldots,x_n) \geq max(x_1,x_2,\ldots,x_n) \tag{16}$$

holds for all $x_i \in [0,1]$. A function f generates an *identity modifier* $F_0 = f(x,x,\ldots,x)$ if for all $x_1,x_2,\ldots,x_n \in [0,1]$ the condition

$$min(x_1,x_2,\ldots,x_n) \leq f(x_1,x_2,\ldots,x_n) \leq max(x_1,x_2,\ldots,x_n) \tag{17}$$

holds for all $x_i \in X$.

It is easily shown that the formulas (15), (16), and (17) are in accordance with those in Def.1 when we replace x with the membership function of a fuzzy set to be modified. Thus we can use these formulas as the conditions for n-ary functions for generating modifiers.

Example 1 *The formula*

$$f(x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} x_i, \qquad \forall i, x_i \in [0,1], \tag{18}$$

generates a substantiating modifier

$$F_{n-1}(\mu(x)) = (\mu(x))^n, \qquad \forall x \in X, \tag{19}$$

because it clearly satisfies the condition (15), i.e. $f(x_1,x_2,\ldots,x_n) \leq min(x_1,x_2,\ldots,x_n)$*. The bigger n is the more substantiating modifier we have. Thus we can have a graded system of modifiers. Especially, if* $n = 1$*, we have the identity modifier* $F_0 = \mu$*, that have no substantiating effect.*

Example 2 *The formula*

$$f(x_1,x_2,\ldots,x_n) = 1 - \prod_{i=1}^{n}(1-x_i), \qquad \forall i, x_i \in [0,1], \tag{20}$$

generates a weakening modifier

$$H_{n-1}(\mu(x)) = 1-(1-\mu(x))^n, \qquad \forall x \in X \tag{21}$$

because it clearly satisfies the condition (16). To see this, let a delivery of values from the interval $[0,1]$ *be such that* $x_k, 1 \leq k \leq n$*, has the greatest value. In this situation we can write* $max(x_1,x_2,\ldots,x_n) = x_k$*. Thus we have* $1-x_k \geq 1-x_i, 1 \leq i \leq n$*, and this implies*

$$(1-x_k)^n \geq \prod_{i=1}^{n}(1-x_i)$$

which implies

$$1 - \prod_{i=1}^{n}(1-x_i \geq 1-(1-x_k)^n.$$

From this it clearly follows that $1-(1-x_k)^n \geq 1-(1-x_k) = x_k = max(x_1,x_2,\ldots,x_n)$ *by the suppoaition of the delivery of values. The special case* $n = 1$ *gives the idetity modifier* $H_0(\mu(x)) = \mu(x) \forall x \in X$*, as it should be.*

Example 3 *In addition to the special cases of previous examples,consider some generators for identity modifier. The formula*

$$f(x_1,x_2,\ldots,x_n) = \frac{1}{n}(x_1+x_2+\ldots+x_n), \tag{22}$$

generates identity modifier, because (14) holds clearly.

Another way for generating identity modifier is to use the function

$$f(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} \lambda_i x_i. \tag{23}$$

It is easy to show that this function satisfies the condition (14).

Also $max(x_1,x_2,\ldots,x_n)$ *and* $min(x_1,x_2,\ldots,x_n)$ *generate identity modifiers, because the operators max and min do not have any modifying effect.*

According to Def.2, the *dual* of a modifier F is defined by the condition

$$F^*(x) = n(F(n(x))) \tag{24}$$

where n is a negation function. This also means that if F is substantiating then F^* is weakening, and if H is weakening then H^* is substantiating, by Proposition 1.

Example 4 *The modifiers given in Examples 1 and 2 are duals of each others when* $\forall x \in X, n(\mu(x)) = 1 - \mu(x)$. *The modifiers (19) and (21) are basing on extensions of the t-norm* algebraic product *and the t-conorm* algebraic sum, *respectively.*

3 Some Concluding Remarks

One purpose for studying modifiers is to create some concrete tools for manipulating fuzzy numbers so that we can have arithmetic operations easily used. However, these operations should be in accordance with those defined by extension principle. Also the study of logical systems of modifiers is very interesting. From this study we can draw connections to topological properties of fuzzy systems (see e.g. Kortelainen's paper [3] and his other papers, too).

According to the substance itself, n-ary functions being extensions of some Archimedean t-norms and t-conorms are very interesting for generators of modifiers, as we already had a short view in the form of Examples 1 – 4 above. It is well known that Archimedean t-norms and corresponding t-conorms have modifying effects (see e.g. Mattila [8]).

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