Inconsistencies in Linguistic Descriptions from the Point of View of Fuzzy Logic

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Abstract

An important notion of inconsistency occurring in the theory of IF-THEN rules is discussed. We propose a definition of it in the frame of fuzzy logic deduction and show some properties.

Keywords: IF-THEN rules, deduction, inconsistency.

1 Introduction

In this contribution we will discuss one important notion occurring in the theory of linguistic descriptions, i.e. sets of IF-THEN rules. The situation we want to study is as follows: there are several IF-THEN rules with identical (or similar) antecedents and different consequents. We call such a linguistic description *inconsistent*, because given an observation, several rules can be used to deduce conclusions which are incompatible. For example, let a linguistic description which describes some control process include two IF-THEN rules

$$\begin{aligned} \mathcal{R}_1 &:= \text{IF } X_1 \text{ is } \mathit{medium} \text{ AND } X_2 \text{ is } \mathit{big} \\ &\quad\ \text{THEN } Y \text{ is big}, \\ \mathcal{R}_2 &:= \text{IF } X_1 \text{ is } \mathit{medium} \text{ AND } X_2 \text{ is } \mathit{big} \\ &\quad\ \text{THEN } Y \text{ is } \mathit{small}, \\ &\quad\ \ldots \end{aligned}$$

where X_1, X_2 and Y are variables denoting temperature, pressure and control action, respectively. Then, when medium temperature and big pressure are measured, both rules $\mathcal{R}_1$ and $\mathcal{R}_2$ can be used and we arrive at the conclusion that the control action should be at the same time *small* and *big*.

In our analysis of this situation we start from the observation that a linguistic description describes some functional dependence and that variable Y can assume only one value at a given time. Our definition is motivated by the classical requirement which a relation has to fulfil in order to be a function.

The basis for our analysis is *fuzzy logic in narrow sense with evaluated syntax* [8]. We consistently distinguish three levels of study: linguistic, syntactic and semantic. The meaning of a linguistic expression $\mathcal{A}$ (called *intension*) is expressed on syntactic level by a set of evaluating formulas

$$\mathrm{Int}(\mathcal{A}) = \mathbf{A}_{\langle x\rangle} = \left\{ {}^{a_t}\!/A_x[t] \mid t \in M, a_t \in L \right\}$$

where A is a formula with one free variable assigned to the linguistic expression $\mathcal{A}$, M is a set of closed terms – names for real numbers from interval $[0,1]$, L is a set of truth values, identified with the set $[0,1]$, and $A_x[t]$ is the closed formula arisen from formula A with the term t is substituted for the variable x.

Meanings of the linguistic description and the observation are represented on syntactic level by formal fuzzy theories T_I and T', respectively. Another theory T_D is obtained as a union of T_I and T'. The conclusion is then obtained by means of a formal proving inside the theory T_D. On semantic level, the concept of *possible world* is introduced as a special structure which support is a real interval. Then the meaning of a linguistic expression on semantic level is the satisfaction fuzzy set

$$\mathrm{Ext}_{\mathcal{V}}(\mathcal{A}) = \left\{ {}^{\mathcal{V}(A_x[\mathbf{v}])}\!/v \mid v \in V \right\}$$

where $\mathcal{V}$ is a possible world, V is its support, $\mathbf{v}$ is the closed term – name of the real number v on syntactic level and $\mathcal{V}(A_x[\mathbf{v}])$ truth value of formula $A_x[\mathbf{v}]$ in the possible world $\mathcal{V}$.

2 Preliminaries

2.1 Logical preliminaries

The formal system of *fuzzy logic in narrow sense with evaluated syntax* (FLn) we are working in is based on Łukasiewicz MV-algebra of truth values

$$\mathcal{L} = \langle L, \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$$

where the set of truth values $L = [0,1]$, and $\otimes, \oplus, \neg$ are Łukasiewicz t-norm, Łukasiewicz t-conorm and the involutive negation $\neg x = 1 - x$, respectively.

Let $A(x_1,\dots,x_n)$ be a formula and $t_1,\dots,t_n$ be terms substitutable into A for the variables $x_1,\dots,x_n$, respectively. By $A_{x_1,\dots,x_n}[t_1,\dots,t_n]$, we denote an instance of A resulting from it when replacing all the free occurrences of the variables $x_1,\dots,x_n$ by the respective terms $t_1,\dots,t_n$.

A fuzzy theory T is a fuzzy set of formulas $T \underset{\sim}{\subset} F_{J(T)}$ ($J(T)$ is the predicate language of the theory T and $F_{J(T)}$ is the set of all well-formed formulas of $J(T)$) given by the triple $T = \langle \text{LAx}, \text{SAx}, R \rangle$ where $\text{LAx} \underset{\sim}{\subset} F_{J(T)}$ is a fuzzy set of logical axioms, $\text{SAx} \underset{\sim}{\subset} F_{J(T)}$ is a fuzzy set of special axioms and R is a set of inference rules which includes the rules modus ponens (r_{MP}), generalization (r_G) and logical constant introduction (r_{LC}) (see [8], Section 4.3.1).

The semantics is defined by the generalization of the classical definition of the semantics of predicate logic. A *structure* for the language J is

$$\mathcal{V} = \langle V, f_{\mathcal{V}}, \dots, P_{\mathcal{V}}, \dots, u_{\mathcal{V}}, \dots \rangle \qquad (1)$$

where $f_{\mathcal{V}} : V^n \longrightarrow V$ are n-ary functions on V assigned to the function symbols $f \in J$, $P_{\mathcal{V}} \underset{\sim}{\subset} V^n$ are n-ary fuzzy relations on V assigned to the predicate symbols $P \in J$ and $u_{\mathcal{V}} \in V$ are designated elements assigned to the object constants $\mathbf{u} \in J$. If the concrete symbols $f_{\mathcal{V}}, P_{\mathcal{V}}, u_{\mathcal{V}}, \dots$ are unimportant for the explanation then we will simplify (1) only to $\mathcal{V} = \langle V, \dots \rangle$.

We say that the structure $\mathcal{V}$ is a *model* of the fuzzy theory T and write $\mathcal{V} \models T$ if $\text{SAx}(A) \leq \mathcal{V}(A)$ holds for every formula $A \in F_{J(T)}$.

The concept of the provability degree as a generalization of the classical provability $T \vdash_a A$ and truth degree $T \models_a A$ can be introduced (for the precise definitions and a lot of properties of them — see [8]). Let us stress that the provability degree coincides with the truth degree due to the completeness theorem.

Theorem 1 (Completeness)

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A$$

holds for every formula $A \in F_{J(T)}$ *and every consistent fuzzy theory* T.

Fuzzy theory with fuzzy equality contains in its language binary predicate $\approx$ which should fulfil the relevant axioms (cf. [6]).

2.2 Linguistic preliminaries

A general surface structure of fuzzy IF-THEN rule is

$$\text{IF } \langle\text{noun}\rangle_1 \text{ is } \mathcal{A} \text{ THEN } \langle\text{noun}\rangle_2 \text{ is } \mathcal{B}. \qquad (2)$$

It is a conditional statement characterizing relation between linguistic expressions of the form

$$\langle\text{noun}\rangle \text{ is } \mathcal{A}. \qquad (3)$$

We will call expressions (3) the *linguistic predications*. Examples of such predications are *temperature is quite high, angle of the wheel is negative very big, (breaking) force is more or less small*, etc.

For the purpose of modeling using fuzzy logic, we usually are not interested in the objects denoted by nouns occurring in the linguistic predications. In the practice, they are replaced by numbers. Therefore, we replace $\langle$noun$\rangle$ in (2) by some variable $X, Y, \dots$ etc. Consequently, the general surface structure of fuzzy IF-THEN rule considered further is

$$\text{IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}. \qquad (4)$$

A special case of the expressions occurring in the linguistic predications are *evaluating linguistic expressions* (cf. [8, 5]). Among them, we distinguish *atomic evaluating expressions* which include any of the adjectives *small, medium,* or *big.*

Simple evaluating expressions are expressions of the form

$$\langle\text{linguistic hedge}\rangle\langle\text{atomic evaluating expression}\rangle.$$

The linguistic hedges are special adjectives modifying the meaning of the adjectives before which they stand. In general, we speak about the linguistic hedges with *narrowing effect* (*very, highly,* etc.) and those with *widening effect* (*more or less, roughly,* etc.).

If $\mathcal{A}$ in (3) is an evaluating expression then (3) is called the *evaluating linguistic predication*. If $\mathcal{A}$ is simple then (3) is called the *simple evaluating predication*. Examples of simple evaluating predications are, e.g. *temperature is very high* (here *high* is taken instead of *big*), *pressure is roughly small*, etc.

Definition 1
A *linguistic description* is a finite set $\mathcal{LD}^I = \{\mathcal{R}_1^I, \mathcal{R}_2^I, \ldots, \mathcal{R}_r^I\}$ of the conditional clauses

$$\mathcal{R}_i^I := \text{IF } \mathcal{A}_i \text{ THEN } \mathcal{B}_i, \quad i = 1, 2, \ldots, r \tag{5}$$

where $\mathcal{A}_i$, $\mathcal{B}_i$ are evaluating predications. If all evaluating predications $\mathcal{A}_i$, $\mathcal{B}_i$ are simple then also the linguistic description $\mathcal{LD}^I$ is called simple.

2.3 The meaning of linguistic expressions

A linguistic expression may in general be understood as a name of some property. In the linguistic theory, we speak about its *intension* (instead of property named by it). Furthermore, we have to consider a *possible world* (cf. [10]), which can be informally understood as "a particular state of affairs". For us, the possible world is a set of objects, which may carry the properties in concern. Hence, the intension of the linguistic expression determines in each possible world its *extension*, i.e. a grouping of objects having the given property. Since there can exists infinite number of possible worlds, one intension may lead to a class of extensions.

We will formalize these concepts using the means of predicate FLn with evaluated syntax. The level of formal syntax is identified with the syntax of FLn and the semantic level is identified with the semantics of FLn. In the sequel, we suppose some fixed predicate language J.

To define the mathematical model of the *intension* of a linguistic expression $\mathcal{A}$, we start by assigning some formula $A(x) \in F_J$ to $\mathcal{A}$. However, this is not sufficient since this does not grasp inherent vagueness of the property represented by $\mathcal{A}$. This can be accomplished in FLn using the concept of *evaluated formula*. Namely, if $A(x)$ is a formula with one free variable then the evaluated formula $a/A_x[t]$ means that some object represented by the term t has the property A in the degree at least $a \in L$.

The *extension* is characterized on the semantic level, which is identified with the semantics of FLn. Hence, the concept of *possible world* is understood as a special structure $\mathcal{V}$ for J

$$\mathcal{V} = \langle V, P_{\mathcal{V}}, \ldots, \rangle.$$

Definition 2
Let $\mathcal{A}$ be a natural language expression and let it be assigned a formula $A(x)$.

(i) The intension of $\mathcal{A}$ is a set of evaluated formulas (also called *multiformula*)

$$\text{Int}(\mathcal{A}) = \mathbf{A}_{\langle x \rangle} = \left\{ a_t/A_x[t] \,\middle|\, t \in M, a_t \in L \right\}. \tag{6}$$

(ii) The extension of $\mathcal{A}$ in the possible world $\mathcal{V}$ is the satisfaction fuzzy set

$$\text{Ext}_{\mathcal{V}}(\mathcal{A}) = \left\{ \mathcal{V}(A_x[\mathbf{v}])/v \,\middle|\, v \in V \right\}. \tag{7}$$

3 Theories of evaluating expressions

Due to the lack of space, we can only briefly summarize main ideas of our approach (for details, see [2]). We propose to capture the meanings of the class of simple evaluating linguistic expressions from Subsection 2.2 by formal theory T^{ev} called the *theory of evaluating expressions*. For every simple evaluating expression $\mathcal{A}_i$ we add unary predicate symbol G_i into the language $J(T^{ev})$ of theory T^{ev}. Then, an intension of $\mathcal{A}_i$ in T^{ev} is

$$\text{Int}(\mathcal{A}_i) = \left\{ \tilde{\alpha}_{G_i}(t)/G_{i,x}[t] \mid t \in M, T^{ev} \vdash_{\tilde{\alpha}_{G_i}(t)} G_{i,x}[t] \right\} \tag{8}$$

where $\tilde{\alpha}_{G_i} : M \longrightarrow L$ is a function called *intensional mapping*. We propose several axioms which such a theory T^{ev} should fulfil in order to characterize properly the meanings of simple evaluating linguistic expressions (see [2], Definition 4). We e.g. require that functions $\tilde{\alpha}_{G_i}$ should be unimodal.

Because it holds that the truth degree of a formula A in a model $\mathcal{V}$ of a theory T (denoted by $\mathcal{V}(A)$) is, in general, greater or equal than its provability degree in T, it can be advantageous to restrict the range of truth degrees of formulas $G_x[t]$ from above by introducing the axioms of the form

$$\left\{ \tilde{\beta}_G(t)/\neg G_x[t] \mid t \in M \right\} \tag{9}$$

where $\tilde{\beta}_G(t) : M \longrightarrow L$, and $\tilde{\beta}_G(t) \le \neg\tilde{\alpha}_G(t)$ holds for all $t \in M$. We call the resulting theory the *extended theory of evaluating expressions* and denote it by T^{evx}.

Given a linguistic description $\mathcal{LD}^I$ and theories T_1^{ev} and T_2^{ev} (or T_1^{evx} and T_2^{evx}) for antecedent and consequent part of IF-THEN rules, respectively, we can construct a theory T_I called the *theory of linguistic description*. Its axioms have the form

$$\tilde{\alpha}_{A_i \Rightarrow B_i}(t,s) \big/ A_{i,x}[t] \Rightarrow B_{i,y}[s] \tag{10}$$

where A_i, B_i are formulas corresponding to simple evaluating expressions $\mathcal{A}_i$, $\mathcal{B}_i$, respectively, from the i-th IF-THEN rule IF X is $\mathcal{A}_i$ THEN Y is $\mathcal{B}_i$, and $t \in M_1$, $s \in M_2$, where M_1, M_2 are sets of closed terms from the language of theories T_1^{ev}, T_2^{ev}, respectively.

4 Fuzzy logic deduction

The basic schema of fuzzy logic deduction is the following: We have a fuzzy theory T_I composed of implications of the form (10) and a fuzzy theory T' which represents an observation. From these theories we form a theory $T_D = T_I \cup T'$. The theory T_I expresses a relationship between antecedent and succedent variables. The general form of T' is

$$T' = \left\{ \mathbf{A}'_i \mid i \in I \right\} \tag{11}$$

where $I \subseteq \{1, 2, \ldots, r\}$ and

$$\mathbf{A}'_i = \left\{ \tilde{\alpha}'_{A_i}(t) \big/ A_{i,x}[t] \mid t \in M_1 \right\}. \tag{12}$$

If the theory $T_D = T_I \cup T'$ is consistent then we can derive a *conclusion*

$$\mathbf{B}' = \{\mathbf{B}'_i \mid i \in K\} \tag{13}$$

where

$$\mathbf{B}'_i = \left\{ \tilde{\alpha}'_{B_i}(s) \big/ B_{i,y}[s] \mid s \in M_2 \right\} \tag{14}$$

and $\tilde{\alpha}'_{B_i}(s) = c$ iff $T_D \vdash_c B_{i,y}[s]$. The theory T', in general, can contain several multiformulas $\mathbf{A}'_i$, and hence, the conclusion $\mathbf{B}'$ may be composed of several parts. The index sets I and K are identical if $i \neq j$ implies $A_i \neq A_j$ for all $i, j \in I$. Note that T' can be empty. Then we put $\mathbf{B}' = \emptyset$, i.e. we obtain an empty conclusion. This is in accordance with the principle of logic deduction, saying that the deduction is performed only when there is enough evidence for doing it.

In the most frequent case, the theory T' represents a single observation. This can be either a crisp number or a linguistic expression, which characterizes (precisely or vaguely) a position on an ordered scale. Therefore, the theory T' cannot be completely arbitrary and should fulfil some special conditions (see [2]).

Theorem 2
Let $\mathcal{LD}^I$ be a simple linguistic description and the theory $T_D = T_I \cup T'$ be constructed as above. Then it is consistent and we may derive a conclusion $\mathbf{B}' = \{\mathbf{B}'_i \mid i \in K\}$ where the intensions $\mathbf{B}'_i$ are

$$\mathbf{B}'_i = \left\{ \tilde{\alpha}'_{B_i}(s) \big/ B_{i,y}[s] \mid s \in M_2,\ i \in K \right\} \tag{15}$$

where

$$\tilde{\alpha}'_{B_i}(s) = \bigvee_{t \in M_1} \left(\tilde{\alpha}'_{A_i}(t) \otimes \tilde{\alpha}_{A_i \Rightarrow B_i}(t,s) \right) \tag{16}$$

and all $\tilde{\alpha}'_{B_i}(s)$ in $\mathbf{B}'_i$, $i \in K$ are maximal.

PROOF: The proof proceeds in a similar way as the proof of Theorem 6.1 in [8], page 249. □

5 Inconsistencies in linguistic descriptions

It can be shown (see [2], Theorem 5) that the theory T_I of a linguistic description $\mathcal{LD}^I$ is consistent. It is a natural result, because it is not possible to derive a contradiction from implications only. However, there are some linguistic descriptions which behave "inconsistently." By this is meant that for some observation there are derived conclusions, whose interpretations are at the same time "small" and "big". A typical example of "inconsistent" linguistic description is

$\mathcal{R}_1 :=$ IF X is medium THEN Y is big,
$\mathcal{R}_2 :=$ IF X is medium THEN Y is small,
...

Here, if the observation is $\mathcal{A}' = \mathit{medium}$, we cannot decide between rules $\mathcal{R}_1$ and $\mathcal{R}_2$ and we should use both rules and, consequently, deduce that Y is big and Y is small at the same time. If we understand Y as the (precisely of imprecisely known) value of some variable, then there is a contradiction in the fact, that Y is at the same time "small" and "big".

To characterize this type of inconsistency of the theory $T_D = T' \cup T_I$, we propose to capture the fact that Y is a variable and therefore cannot have more than one value at a given time, (or, for one observation) by the provability of some formulas. We use fuzzy equality predicate $\approx$ to express this property. Let us denote by $T_D^{\approx}$ the fuzzy theory

$$T_D^{\approx} = T_D \cup T_2^{\approx} \tag{17}$$

where

$$T_2^{\approx} = \{d(s_1,s_2)/s_1 \not\approx s_2 \mid s_1,s_2 \in M_2\} \tag{18}$$

where $s_1 \not\approx s_2$ is an abbreviation for $\neg(s_1 \approx s_2)$ and $d(s_1,s_2)$ is a provability degree of formula $s_1 \not\approx s_2$ in some theory T_2, i.e. $d(s_1,s_2) = d$ iff $T_2 \vdash_d s_1 \not\approx s_2$, and T_2 is some subtheory of the theory of evaluating expressions $T_{2\approx}^{evx}$, i.e. the extended theory of evaluating expressions with fuzzy equality.

Definition 3
The theory $T_D^{\approx} = T' \cup T_I \cup T_2^{\approx}$ is $\approx$*-inconsistent in the degree* κ if for some theory T' which represents a single observation (see [2], Definition 9) the theory $T_D^{\approx}$ proves

$$T_D^{\approx} \vdash_\kappa (\exists y_1)(\exists y_2)\ (y_1 \not\approx y_2)\ \&\ B_1(y_1)\ \&\ B_2(y_2). \tag{19}$$

If $\kappa = 1$ then $T_D^{\approx}$ is called $\approx$*-inconsistent*.

The previous definition is motivated by the classical requirement which a relation has to fulfil in order to be a function:

$$(\forall x)(\forall y)(\forall z)\ (y = f(x)\ \&\ z = f(x)) \Rightarrow (y = z).$$

Negation of this formula is logically equivalent to

$$(\exists x)(\exists y)(\exists z)\ (y = f(x)\ \&\ z = f(x)\ \&\ z \neq y). \tag{20}$$

Because $B_1(y_1)$ and $B_2(y_2)$ from (19) express a value of variable Y for some single observation characterized by the theory T', formula (19) expresses in our formalism the same property as formula (20) in the classical case. Let us stress that the above-defined notion of $\approx$-inconsistency is dependent on the intensional mappings $\tilde{\alpha}_G$ of predicate symbols G. This is the reason why we defined the inconsistency of theory $T_D^{\approx}$ and not the inconsistency of the linguistic description $\mathcal{LD}^I$. If we include into the theory $T_2^{\approx}$ axioms of the form $d(s_1,s_2)/s_1 \not\approx s_2$ with degrees $d(s_1,s_2)$ computed from the whole theory T^{evx}, i.e. the theory which describes the intensions of the whole set of simple evaluating expressions, then the definition of $\approx$-inconsistency can become too sensitive. It means that $T_D^{\approx}$ can prove that $x \not\approx y$ for x and y which are "close" to each other (in standard metric on [0,1]) but are distinguished by some predicate G_i in the sense that

$$T^{evx} \vdash G_i(x) \not\Leftrightarrow G_i(y) \tag{21}$$

and, consequently, $T_D^{\approx} \vdash x \not\approx y$. To be able to prove formulas such as (21), we have to work with the extended theory of evaluating expressions. As a possible solution we can use some subtheory T_2 of T^{evx}, i.e. a theory which includes axioms of the form (8) and (9) for some subset of the set of simple evaluating expressions. The structure of this set (see Subsection 2.2) can give us a clue which evaluating expressions should be chosen. We can choose e.g. predicates which model atomic evaluating expressions *small*, *medium* and *big* or predicates which model some wider evaluating expressions for every subset of evaluating expressions with the same atomic one, such as *roughly small*, *roughly medium* and *roughly big*.

Lemma 1
If there are closed terms s_1 and $s_2 \in M_2$ such that $T_D \vdash B_{i,y}[s_1]$ and $T_D \vdash B_{j,y}[s_2]$ and

$$T_2 \vdash B_{k,y}[s_1] \not\Leftrightarrow B_{k,y}[s_2] \tag{22}$$

for some $i, j \in 1,2,\ldots,r$ and some $k \in K$, then the theory $T_D^{\approx}$ is $\approx$-inconsistent.

PROOF: Omitted due to the lack of space. □

Theorem 3
Let $\mathcal{LD}^I$ be a simple linguistic description which includes rules $\mathcal{R}_j$ and $\mathcal{R}_k$ such that $\mathcal{R}_j :=$ IF $\mathcal{A}$ THEN $\mathcal{B}_j$ and $\mathcal{R}_k :=$ IF $\mathcal{A}$ THEN $\mathcal{B}_k$. Let the theories of evaluating expressions T_1^{evx} and T_2^{evx} are such that the theory $T_D^{\approx} = T' \cup T_D \cup T_2$ (17), where T' has the form $T' = \{\tilde{\alpha}'_A(t)/A_x[t] \mid t \in M_1\}$ and it holds that there is some $t_0 \in M_1$ such that $\tilde{\alpha}'_A(t_0) = \tilde{\alpha}_A(t_0) = 1$, proves for some $s_1, s_2 \in M_2$ that $T_D^{\approx} \vdash B_{j,y}[s_1]$, $T_D^{\approx} \vdash B_{k,y}[s_2]$, and

$$T_2 \vdash (\exists y)(B_j(y) \not\Leftrightarrow B_k(y)) \tag{23}$$

where $T_2 \precsim T_{2\approx}^{evx}$. Then the theory $T_D^{\approx}$ is $\approx$-inconsistent.

PROOF: We have to show that from (23) it follows that there exist closed terms s_a and s_b from M_2 such that $T_2 \vdash s_a \not\approx s_b$ (and, consequently, $T_D^{\approx} \vdash s_a \not\approx s_b$) and in the same time that $T_D^{\approx} \vdash B_{j,y}[s_a]$ and $T_D^{\approx} \vdash B_{k,y}[s_b]$.

From (23) and the Completeness Theorem it follows that for some $s \in M_2$ either a) $T_2 \vdash B_{j,y}[s]$ and $T_2 \vdash \neg B_{k,y}[s]$, or b) $T_2 \vdash \neg B_{j,y}[s]$ and $T_2 \vdash B_{k,y}[s]$. Suppose a). Then for some $s_a, s_b \in M_2$ either a$_1$) $T_2 \vdash B_{j,y}[s_a]$, $T_2 \vdash B_{k,y}[s_b]$ and $T_2 \vdash s_a \leq s$, $T_2 \vdash s < s_b$, or a$_2$) $T_2 \vdash B_{j,y}[s_a]$, $T_2 \vdash B_{k,y}[s_b]$ and $T_2 \vdash s \leq s_a$, $T_2 \vdash s_b < s$. Suppose a$_1$). From assumptions a) and a$_1$) it also follows that

$$T_2 \vdash B_{k,y}[s_b] \not\Leftrightarrow B_{k,y}[s],$$

and, due to provable inequalities $s_a \leq s$, $s < s_b$ and unimodality of functions $\tilde{\alpha}_{B_k}$ and $\tilde{\beta}_{B_k}$, also

$$T_2 \vdash B_{k,y}[s_a] \not\Leftrightarrow B_{k,y}[s_b]$$

(see Figure 1). It can be deduced that $T_2 \vdash s_a \not\approx s_b$.

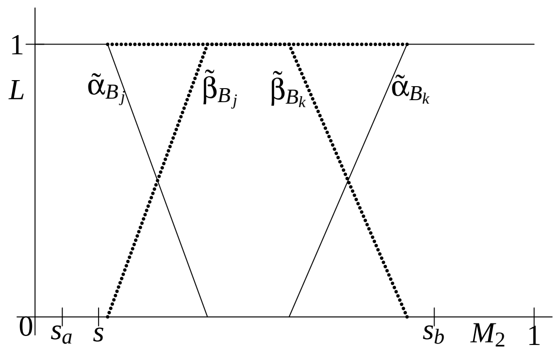

Figure 1: Situation a_1) from the proof of Theorem 3.

From the definition of theory $T_D^{\approx}$ it follows that also $T_D^{\approx} \vdash (s_a \not\approx s_b)$. Because $T_2 \vdash B_{j,y}[s_a]$, $T_2 \vdash B_{k,y}[s_b]$, it holds that $\tilde{\alpha}_{B_j}(s_a) = \tilde{\alpha}_{B_k}(s_b) = 1$. It can easily be deduced that $\tilde{\alpha}'_{B_j}(s_a) = \tilde{\alpha}'_{B_k}(s_b) = 1$. Hence, we have shown that $T_D^{\approx} \vdash B_{j,y}[s_a]$, $T_D^{\approx} \vdash B_{k,y}[s_b]$ and $T_D^{\approx} \vdash s_a \not\approx s_b$, and it follows (using the fact that $A_x[t] \Rightarrow (\exists x)A(x)$ is provable in fuzzy predicate calculus and the Completeness Theorem) that

$$T_D^{\approx} \vdash (\exists y_1)(\exists y_2)\ (y_1 \not\approx y_2) \,\&\, B_j(y_1) \,\&\, B_k(y_2),$$

i.e. the theory $T_D^{\approx}$ is $\approx$-inconsistent. The cases b) and a$_2$) can be proved analogously. □

6 Conclusion

We defined in this paper $\approx$-inconsistency and proved a theorem showing that linguistic descriptions which include two rules with identical antecedents and different succedents are indeed $\approx$-inconsistent. The alternative definition of the inconsistency of linguistic descriptions will be the topic of a subsequent paper.

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