Cancellative t-norms

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Abstract

The structure of cancellative t-norms is studied. It is shown that a cancellative t-norm is generated if and only if it has no anomalous pair, and then it is Archimedean. Moreover, if it is also continuous in point (1,1) it is isomorphic to the product t-norm. Several examples of non-generated cancellative t-norms are also given.

Keywords: Additive generator, anomalous pair, Archimedean semigroup, cancellativity.

1 Introduction

The t-norms generated by a continuous additive generator are exactly the continuous Archimedean t-norms. The t-norms generated by non-continuous generators are not yet fully characterized. In the case of non-continuous generators there is a problem with ensuring associativity. In the book [6] there a sufficient condition for an additive generator to generate a t-norm was given : the range of the generator is relatively closed under addition. Later there were done some generalizations of this condition in [9]. However, the class of all generated t-norms is not yet fully characterized. The aim of this paper is to examine cancellative t-norms and their relationship to additive generators.

2 Cancellative t-norms and additive generators

Definition 1
A t-norm T is called cancellative if for all $x,y,z \in]0,1]^2$, $y = z$ whenever $T(x,y) = T(x,z)$.

Definition 2
A decreasing function $f : [0,1] \to [0,\infty]$, $f(1) = 0$ is called an additive generator of a t-norm T if the operation given by

$$T(x,y) = f^{(-1)}(f(x)+f(y)),$$

is a t-norm, where $f^{(-1)} : [0,\infty] \to [0,1]$ is the pseudo-inverse of f [5] given by

$$f^{(-1)}(u) = \sup\{x \in [0,1] \mid f(x) > u\}.$$

Definition 3
Let T be a t-norm. Then the pair $(a,b) \in]0,1[^2$, $a > b$, is called an anomalous pair of T if

$$a > b > a_T^{(2)} > b_T^{(2)} > \ldots > a_T^{(n)} > b_T^{(n)} > \ldots,$$

where $a_T^{(1)} = a$ and $a_T^{(n+1)} = T(a, a_T^{(n)})$, $n \in \mathbb{N}$.

Definition 4
A decreasing function $f : [0,1] \to [0,\infty]$, $f(1) = 0$ is called a strong additive generator of a t-norm T if for all $n \in \mathbb{N}$ and $(x_1,\ldots x_n) \in [0,1]^n$

$$T(x_1,\ldots,x_n) = f^{(-1)}(\sum f(x_i)).$$

Following [3] we can derive the next result.

Theorem 1
A cancellative t-norm has a strong additive generator if and only if it has no anomalous pair.

Remark 1
When a t-norm T has a strong additive generator f, then f is also an additive generator of T. Vice-versa, if an additive generator f of a t-norm T has a range which is relatively closed under addition, then it is a strong additive generator of T.

Next we will look at the situation, when the generator has a range which is not relatively closed under addition. Let us first see the example introduced in [8]

Example 1
Let $f : [0,1] \to [0,\infty]$ be given by

$$f = \begin{cases} 3-2x & \text{if } x \in [0, \frac{1}{2}[, \\ 2-2x & \text{else.} \end{cases}$$

Then f is an additive, but not a strong additive generator of the t-norm T given by:

$$T(x,y) = \begin{cases} \frac{1}{2} & \text{if } x \geq \frac{1}{2}, y \geq \frac{1}{2}, \\ & x+y \leq \frac{3}{2}, \\ T_L(x,y) & \text{else,} \end{cases}$$

where T_L is the Łukasiewicz t-norm.

After this example it is clear that not all additive generators are also strong additive generators. This leads us to a question whether there exists a generated cancellative t-norm with an additive generator, which has anomalous pairs.

For a cancellative t-norm T generated by a decreasing function $f : [0,1] \to [0,\infty]$ we get the following result.

Theorem 2
Let T be a cancellative t-norm generated by an additive generator f. Then there exists no anomalous pair of T.

We have the following strengthening of Theorem 1.

Corollary 1
A cancellative t-norm is generated if and only if it has no anomalous pairs.

In fact if a generated t-norm is cancellative its generator necessarily has a range closed under addition.

Theorem 3
Let T be a t-norm generated by an additive generator f. Then T is cancellative if and only if for all $a,b \in [0,1]$ there exists an element $c \in [0,1]$ such that $f(a)+f(b) = f(c)$, i.e., $\mathrm{Ran}(f)+\mathrm{Ran}(f) \subseteq \mathrm{Ran}(f)$.

Corollary 2
A cancellative t-norm, which has no anomalous pairs is continuous in the point (1,1) if and only if it is continuous on the whole unit square.

In [4] it was proved:

Theorem 4
A cancellative Archimedean t-norm is continuous in the point (1,1) if and only if it is continuous on the whole unit square.

Corollary 3
A cancellative t-norm which is continuous in point (1,1) is continuous and Archimedean, i.e., it is a strict t-norm if and only if it has no anomalous pairs.

3 Cancellative t-norms with anomalous pairs

We already know that a cancellative t-norm is generated if and only if it has no anomalous pairs. This shows us that the following t-norms are not generated. The first example of this type was introduced in [1].

Example 2
It is known that each element $x \in]0,1]$ can be expressed by $x = \sum \frac{1}{2^{x_i}}$, where $\{x_i\}_{i\in\mathbb{N}}$ is an increasing sequence of natural numbers $x_i \in \mathbb{N}$. Let the t-norm T be defined for $(x,y) \in]0,1[^2$ by $T(x,y) = \sum \frac{1}{2^{z_i}}$, where $x = \sum \frac{1}{2^{x_i}}$, $y = \sum \frac{1}{2^{y_i}}$, $z_i = x_i + y_i, i \in \mathbb{N}$. Then T is a cancellative Archimedean t-norm, which is neither left- nor right-continuous, such that to every element $a \in]0,1[$ there is a b such that (a,b) is anomalous pair.

Definition 5
Let T be a t-norm. Two elements $x,y \in [0,1]$ are called T-Archimedean equivalent if there is an $n \in \mathbb{N}$, such that $x_T^{(n)} \leq y \leq x$ or $y_T^{(n)} \leq x \leq y$. For each $x \in [0,1]$ the equivalence class I_x containing x is called a T-Archimedean class and the pair $(I_x \mid T|_{I_x^2})$ is a subsemigroup of ([0,1],T) and it is called a T-Archimedean component of T.

Next example, very closely connected to Example 2, was introduced in [7] and studied in [2].

Example 3
Let the t-norm T be defined for $(x,y) \in]0,1]^2$ by $T(x,y) = \sum \frac{1}{2^{z_i}}$, where $x = \sum \frac{1}{2^{x_i}}$, $y = \sum \frac{1}{2^{y_i}}$, $z_i = x_i + y_i - i, i \in \mathbb{N}$. Then T is a non-Archimedean, left-continuous cancellative t-norm with anomalous pairs in every non-trivial T-Archimedean class.

Examples 2 and 3 are closely connected because each non-trivial T-Archimedean component from Example 3 is isomorphic to a t-norm from Example 2 (up to the boundary).

In fuzzy logics and probabilistic metric spaces, left-continuous t-norms play a key role. Suppose that T is a cancellative left-continuous t-norm and denote the class of all such t-norms by $\mathcal{T}$. Then due to Corollary 3, $T \in \mathcal{T}$ is strict (i.e., continuous) if and only if it has no anomalous pair. Therefore any non-continuous $T \in \mathcal{T}$ possesses some anomalous pair. Due to Theorem 4, such a T cannot be Archimedean. Moreover, up to the trivial T-Archimedean classes $I_1 = \{1\}$ and $I_0 = \{0\}$, we conjecture that all other T-Archimedean classes form a sequence $\{]a_{n-1}, a_n]\}_{n\in\mathbb{N}}$, where $a_0 = 0$ and $\{a_n\}_{n\in\mathbb{N}}$ is a strictly increasing sequence, $\lim a_n = 1$, and $T|_{[a_{n-1},a_n]^2}$ is a non-generated cancellative Archimedean left-continuous but not continuous t-subnorm on $[a_{n-1}, a_n]$. The study of the structure of left-continuous cancellative t-norms will be the topic of further investigations.

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