Clarifying Elkan's theoretical result

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Abstract

This paper is devoted to clarify the only theoretical result included in the controversial work of C. Elkan "The paradoxical success of fuzzy logic" (1994), by offering both a short proof of it and a larger one that tries to show the structural reasons of such result.

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1 Introduction

1.1. In [2] Charles Elkan considered a system E_t consisting of a 5–tuple $(X, \cdot, +, ', \equiv)$ and a function $t : X \to [0,1]$ where:

- X is supposed to be a set of assertions;

- the operation $\cdot$ mimics conjunction, $+$ mimics disjunction, $'$ mimics negation, and $\equiv$ mimics logical equivalence;

- the function t verifies $t(a \cdot b) = \min(t(a), t(b))$, $t(a + b) = \max(t(a), t(b))$, $t(a') = 1 - t(a)$ and "If $a \equiv b$, then $t(a) = t(b)$".

Then, provided the law $(a \cdot b')' \equiv b + a' \cdot b'$ holds in such a system, Elkan concluded in [2] that t can only take as much as two values.

1.2. It is clear that the hypothesis that X is a set of assertions is not essential, and Elkan did a tedious proof of his result that contributed to prevent its full understanding. Even more, as Elkan presented the result as a failure of fuzzy logic (where there are always true and false elements), it was not captured that in the 5-tuple the existence of true and false assertions was not supposed. Possibly, this is the reason why Elkan did not consider a law like $a \cdot a' \equiv false$ for all $a \in X$, from which his result would have follow immediately, since it translates into $\min(t(a), 1 - t(a)) = t(false) = 0$ and then $t(a) \in \{0, 1\}$.

1.3. As it is $t(X) \subseteq [0,1]$, there exist numbers $i = \inf t \in [0,1]$ and $s = \sup t \in [0,1]$ such that $i \leq t(x) \leq s$ for all $x \in X$. Of course, numbers i and s can or cannot correspond to values of t, i.e., they can or cannot be the minimum and the maximum of t. In any case, as $i \leq t(a') \leq s$, it is $i \leq 1 - t(a) \leq s$ or $1 - s \leq t(a) \leq 1 - i$ for all $a \in X$. Hence, it is $1 - s \leq i$ and $s \leq 1 - i$, that is, $i + s = 1$, and therefore $i \leq 1/2 \leq s$. In addition, since $t(a') = 1 - t(a)$ for all $a \in X$, the only admissible constant function t is $t(a) = 1/2$ for all $a \in X$.

1.4. It should be pointed out that the equivalence $\equiv$ in E_t is not necessarily the equivalence modulo t in X, defined as $a \approx b$ if and only if $t(a) = t(b)$. In general, relation $\equiv$ is included in $\approx$, but they are not coincidental. Hence, Elkan reached his result by avoiding false elements, true elements and the equivalence $\approx$.

There have been many reactions to the mentioned Elkan's paper, analyzing his result from different points of view. Most of them appeared short after Elkan's ideas were first published (see the reference [2] for 15 different responses), while others have been published more recently (see for example [6], [3],[7]).

In what follows we will first offer a very simple and purely arithmetical proof of Elkan's result, and, afterwards, an algebraic proof that, notwithstanding it is more sophisticated, reflects pretty well where the reason of the result structurally lies.

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2 A direct and simple proof

Given a system E_t, if $(a \cdot b')' \equiv b + a' \cdot b'$ holds for any $a, b \in X$, this means that $t((a \cdot b')') = t(b + a' \cdot b')$, and, therefore, $\max(t(b), 1 - t(a)) = \max(t(b), \min(1 - t(a), 1 - t(b)))$. Taking into account that $1 - t(a) \leq \max(t(b), 1 - t(a))$ and that $\max(t(b), \min(1 - t(a), 1 - t(b))) \leq \max(t(b), 1 - t(b))$, we get $1 - t(a) \leq \max(t(b), 1 - t(b))$, or, equivalently, $\min(t(b), 1 - t(b)) \leq t(a)$ for any $a, b \in X$. But the later means that function t has necessarily a minimum value, let us say i, and that for any $b \in X$ it is $\min(t(b), 1 - t(b)) = i$, from which it follows that for any $b \in X$ it is either $t(b) = i$ (the minimum element of t) or $t(b) = 1 - i$ (the maximum element of t). Note that these two values coincide when t is the constant function given by $t(x) = 1/2$ for any $x \in X$.

The above is a simple proof of Elkan's theorem, confirming that the law $(a \cdot b')' \equiv b + a' \cdot b'$, as many other laws of propositional logic, cannot hold in a multiple-valued logic.

We end this section by illustrating this result with a real example. Let us consider a system $E_t = (X, \cdot, +, ', \equiv)$ where X is the set of assertions $\{\mathcal{X} \text{ is } r : r \in \mathbb{R}\}$ over a universe Ω, and where the operations of the system are defined as follows:

$(\mathcal{X} \text{ is } r_1) \cdot (\mathcal{X} \text{ is } r_2) = (\mathcal{X} \text{ is } \min(r_1, r_2))$
$(\mathcal{X} \text{ is } r_1) + (\mathcal{X} \text{ is } r_2) = (\mathcal{X} \text{ is } \max(r_1, r_2))$
$(\mathcal{X} \text{ is } r)' = (\mathcal{X} \text{ is } -r)$
$(\mathcal{X} \text{ is } r_1) \equiv (\mathcal{X} \text{ is } r_2)$ if and only if $r_1 = r_2$

Now, let $t : X \to [0, 1]$ be defined as $t(\mathcal{X} \text{ is } r) = 1/(1 + exp(-r))$. This function, which is continuous, strictly increasing and has a supremum and an infimum but has neither a maximum nor a minimum, is closely related to the well-known logistic function used as an activation function in artificial neural networks. In our context, it verifies, in addition, all the requirements stated in [2] and repeated in the introduction:

(i) $t[(\mathcal{X} \text{ is } r_1) \cdot (\mathcal{X} \text{ is } r_2)] = t[\mathcal{X} \text{ is } \min(r_1, r_2)]$
$= 1/(1 + exp(-\min(r_1, r_2)))$
$= \min[t(\mathcal{X} \text{ is } r_1), t(\mathcal{X} \text{ is } r_2)]$

(ii) $t[(\mathcal{X} \text{ is } r_1) + (\mathcal{X} \text{ is } r_2)] = t[\mathcal{X} \text{ is } \max(r_1, r_2)]$
$= 1/(1 + exp(-\max(r_1, r_2)))$
$= \max[t(\mathcal{X} \text{ is } r_1), t(\mathcal{X} \text{ is } r_2)]$

(iii) $t[(\mathcal{X} \text{ is } r)'] = t(\mathcal{X} \text{ is } -r) = 1 - t(\mathcal{X} \text{ is } r)$, since
$t(\mathcal{X} \text{ is } -r) = 1/(1 + exp(r))$
$= exp(-r)/(exp(-r) + 1) = 1 - t(\mathcal{X} \text{ is } r)$

(iv) Finally, from the definition of $\equiv$, together with the continuity and strict monotonicity of t, it follows that if $(\mathcal{X} \text{ is } r_1) \equiv (\mathcal{X} \text{ is } r_2)$ then $t(\mathcal{X} \text{ is } r_1) = t(\mathcal{X} \text{ is } r_2)$.

It is easy to see that, under these conditions, Elkan's law, that may be written as $-\min(r_1, -r_2) = \max(r_2, \min(-r1, -r2))$, cannot hold for any r_1, r_2. Indeed, it suffices to take, for example, values r_1, r_2 such that $r_1 < r_2 < 0$. Notice that, in this example, equivalences $\equiv$ and $\approx$ are coincidental.

3 An indirect algebraic proof

In this section we will show how Elkan's result can be indirectly proved by applying to the quotient set X/t some general results regarding lattices and Boolean algebras. We first deal with what we will call *De Morgan lattices*, that is, structures $(L, \cdot, +, ')$ such that:

1. $(L, \cdot, +, ')$ is a lattice.
2. $(L, \cdot, +, ')$ is a *pseudo-complemented* lattice, i.e., $' : L \to L$ is a unary operation, called a *pseudo-complement*, verifying:

   2.1. For any $a, b \in L$, if $a \leq b$ then $b' \leq a'$.
   2.2. For any $a \in L$, $(a')' = a$.
3. The operations $\cdot$ and $+$ verify the *distributivity laws*, i.e., for any $a, b, c \in L$ it is $a + (b \cdot c) = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
4. The operations $+, \cdot$ and $'$ verify the *De Morgan* or *duality laws*, i.e., for any $a, b \in L$, $(a + b)' = a' \cdot b'$ and $(a \cdot b)' = a' + b'$.

We first recall the following elementary result:

Lemma 3.1 *If* $(L, \cdot, +, ')$ *is a pseudo-complemented lattice, then either L has a minimum 0 and a maximum 1 verifying* $0' = 1$ *or none of them.*

Proof. Let us suppose that L has a minimum element, i.e., there exists $0 \in L$ such that $0 \leq a$ for any $a \in L$. Then, applying property 2.1. of the pseudo-complement operation, it is $a' \leq 0'$ for any $a \in L$. This will be true, in particular, for $a = b'$, so $(b')' \leq 0'$ for any $b \in L$, which is equivalent, due to property 2.2., to $b \leq 0'$ for any $b \in L$. The later means that $0' \in L$ is the maximum element of L. If we suppose that L has a maximum element $1 \in L$, a similar proof shows that then $1' \in L$ is necessarily the minimum element of L. ∎

Note that De Morgan lattices having a minimum and a maximum are precisely the so-called De Morgan algebras. The verification of the law $(a \cdot b')' = b + a' \cdot b'$ in De Morgan algebras was already studied in [5], concluding that the only De Morgan algebras in which this law holds are those that are Boolean algebras. In the following we show that the same result is obtained when considering the more general structure of a De Morgan lattice.

Theorem 3.1 *Let $(L, \cdot, +, ')$ be a De Morgan lattice. Then the law $(a \cdot b')' = b + a' \cdot b'$ holds for any $a, b \in L$ if and only if L is a Boolean algebra.*

Proof.

$\Rightarrow$ Due to the properties of the operations $\cdot, +$ and $'$ in L, the law is equivalent to $a' + b = (a' + b) \cdot (b + b')$, which implies $a' + b \leq b + b'$, and, therefore, $b \cdot b' = (b + b')' \leq (a' + b)' = a \cdot b' \leq a$ for any $a, b \in L$. But this means that L has necessarily a minimum element 0 (and also, due to lemma 3.1, a maximum element 1 such that $0' = 1$), and that $b \cdot b' = 0$ for any $b \in L$ (which implies, by duality, that $b + b' = 1$ for any $b \in L$). Consequently, L is a Boolean algebra.

$\Leftarrow$ If L is a Boolean algebra, it has necessarily a minimum element 0 and a maximum element 1 such that $a \cdot a' = 0$ and $a + a' = 1$ for any $a \in L$, and then the given law is clearly verified, since:

- $(a \cdot b')' = a' + (b')' = a' + b$.
- $b + a' \cdot b' = (b + a') \cdot (b + b') = (b + a') \cdot 1 = b + a' = a' + b$. ∎

Given an Elkan's system E_t, we now consider the quotient set $X/t = \{[a] : a \in X\}$, where for any $a \in X$ it is $[a] = \{x \in X : t(x) = t(a)\}$.

Proposition 3.1 *Given a system E_t, the set $(X/t, \preceq)$, where $[a] \preceq [b]$ if and only if $t(a) \leq t(b)$ for any $a, b \in X$, is a* chain *with associated operations $\cdot, + : X/t \times X/t \to X/t$ given by $[a] \cdot [b] = [a \cdot b]$ and $[a] + [b] = [a + b]$.*

Proof. The relation $\preceq$ is trivially a total order relation on X/t, so $(X/t, \preceq)$ is a chain. In addition, for any $a, b \in X$ it is:

$[a] \cdot [b] = \min([a], [b]) = \{x \in X : t(x) = \min(t(a), t(b)) = t(a \cdot b)\} = [a \cdot b]$

$[a] + [b] = \max([a], [b]) = \{x \in X : t(x) = \max(t(a), t(b)) = t(a + b)\} = [a + b]$ ∎

Proposition 3.2 *Given a system E_t, the chain $(X/t, \cdot, +)$, equipped with the unary operation $'$: $X/t \to X/t$ defined as $[a]' = [a']$ for any $a \in X$ is a De Morgan chain.*

Proof. The operations $+$ and $\cdot$ verify the distributivity laws since any chain is trivially distributive. Therefore, we have to prove that the operation $'$ is a pseudocomplement and that the De Morgan laws hold in this structure:

- for any $a, b \in X$, if $[a] \preceq [b]$ then $[b]' \preceq [a]'$

If $[a] \preceq [b]$, then $t(a) \leq t(b)$, and therefore $1 - t(b) \leq 1 - t(a)$. But this means that $t(b') \leq t(a')$, i.e., $[b'] \preceq [a']$, or, equivalently, $[b]' \preceq [a]'$.

- for any $a \in X$, $([a]')' = [a]$

The above equality is equivalent to $[(a')'] = [a]$, but $t((a')') = 1 - t(a') = 1 - (1 - t(a)) = t(a)$.

- for any $a, b \in X$, $([a] \cdot [b])' = [a]' + [b]'$ and $([a] + [b])' = [a]' \cdot [b]'$

The first equality is equivalent to $[(a \cdot b)'] = [a' + b']$. But the later is obviously true since $t((a \cdot b)') = 1 - t(a \cdot b) = 1 - \min(t(a), t(b)) = \max(1 - t(a), 1 - t(b)) = \max(t(a'), t(b')) = t(a' + b')$. The second equality is proven in a similar way. ∎

Since the structure $(X/t, \cdot, +, ')$ is a De Morgan lattice, we can now apply the result obtained in theorem 3.1 in order to obtain Elkan's theorem:

Theorem 3.2 *Given a system E_t, if $(a \cdot b')' \equiv b + a' \cdot b'$ holds for any $a, b \in X$, then function t can have as much as two values.*

Proof. According to Elkan's definition, if $(a \cdot b')' \equiv b + a' \cdot b'$ then $t((a \cdot b')') = t(b + a' \cdot b')$. But this means that $[(a \cdot b')'] = [b + a' \cdot b']$, or, equivalently, $([a] \cdot [b]')' = [b] + [a]' \cdot [b]'$. We are therefore working in a structure $(X/t, \cdot, +, ')$ which is a De Morgan chain (proposition 3.2) and where the law $([a] \cdot [b]')' = [b] + [a]' \cdot [b]'$ holds for any $a, b \in X$. Then, applying theorem 3.1, X/t must necessarily be a Boolean algebra. Finally, the fact that X/t is both a chain and a Boolean algebra implies that it cannot have more than two elements. Indeed, it is clear that the only Boolean algebra $(L, \cdot, +, ', 0, 1)$ that is a chain is $L = \{0, 1\}$. To see it, let us suppose that there exists $a \in L$ such that

$a \neq 0,1$. Then it is also $a' \in L - \{0,1\}$, since if it was $a' = 0$ it will be $(a')' = a = 0' = 1$, and if it was $a' = 1$ it will be $(a')' = a = 1' = 0$. On the other hand, since L is a chain, it is either $a \leq a'$ or $a' \leq a$. If it is $a \leq a'$, then $a \cdot a = a \leq a' \cdot a = 0$, and therefore $a = 0$, which is absurd. In a similar way, if it was $a' \leq a$, then $a' \cdot a' = a' \leq a \cdot a' = 0$, i.e., $a' = 0$, which is also absurd. (Note that when t is a constant function, i.e, $t = \frac{1}{2}$, the structure X/t has only one class and may not be considered a "standard" Boolean algebra). ∎

4 Final comments

4.1. What Elkan proved is only the negative statement that no system E_t where the law $(a \cdot b')' \equiv b + a' \cdot b'$ $[*]$ holds can be a proper multiple-valued system.

The two proofs presented in this paper show that Elkan's result is only meaningful when function t attains a minimum value. The first proof shows that the one made by Elkan in [2] is unnecessarily long and complicated, since the result follows from an elementary reasoning completely presented in a few lines. The second proof is nothing else than a variant of the proof of a well-known theorem of logic ([4]), and shows how deeply the law $[*]$ is linked with the strong structure of Boolean algebras.

Systems E_t are rather curious "formal systems" (as Elkan called them in [2]), whose actual form (internal laws and inference rules) is unknown and where, in principle, there are neither truths nor falsities. Even more rare is to add to systems E_t a law that, among De Morgan algebras and orthomodular lattices, is only characteristic of Boolean algebras (see [5]).

The logistic function in section 2 shows a system E_t where if t has no minimum (and hence no maximum) the law $[*]$ cannot hold, thus corroborating that $[*]$ is characteristic of bivaluated systems.

4.2. It can be claimed that Elkan did not understood what fuzzy logic is for. Fuzzy logic is for representing knowledge, both precise and imprecise, by means of fuzzy sets. Hence, provided a system E_t had something to do with fuzzy logic, the set $\{a \in X : t(a) \in \{i,s\}\}$ should not be empty and should be a Boolean algebra contained in E_t in which elements a such that $t(a) = i$ are <u>false</u> and elements a such that $t(a) = s$ are <u>true</u>. Any type of fuzzy logic should contain classical bivaluated logic because, for example, experts sometimes use precise rules for describing systems' behaviour.

4.3. Reproducing the results in [2] but supposing that E_t has truths and falsities and that t verifies, for all $a,b \in X$, $t(a \cdot b) = T(t(a),t(b))$, $t(a+b) = S(t(a),t(b))$, and $t(a') = N(t(a))$, with T a continuous t-norm, S a continuous t-conorm and N a strong negation function, it can be shown ([8]) that the law $[*]$ does hold if and only if $T = \varphi^{-1} \circ Prod \circ (\varphi \times \varphi)$, $S = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$ and $N = \varphi^{-1} \circ (1 - Id) \circ \varphi$, where φ is an order automorphism of the unit interval $[0,1]$, $Prod$ is the product t-norm $Prod(x,y) = x.y$ and W^* is the Łukasiewicz t-conorm $W^*(x,y) = \min(1, x+y)$.

That is, there are uncountable many fuzzy logics verifying the law $[*]$, although they do not verify neither the laws of duality nor the law of non-contradiction, but verify the excluded-middle principle. If Elkan had supposed the laws of duality and strong-negation, $[*]$ would have been $a' + b \equiv b + a' \cdot b'$.

4.4. This last comment concerns the structure of $[*]$. In fuzzy logics, $(a \cdot b')'$ is nothing else than a hidden material implication. In fact, within fuzzy set theories, the fuzzy set $(\mu \cdot \sigma')' = N \circ T \circ (\mu \times N \circ \sigma)$ is $S_N \circ (N \circ \mu \times \sigma) = \mu' + \sigma$ by defining the operation $+$ by means of $S_N = N \circ T \circ (N \times N)$. Both in De Morgan and orthocomplemented lattices it is $(a \cdot b')' = a' + b$.

The second member, $b + a' \cdot b'$, is the so-called Dishkant implication $a \rightarrow_D b$ of orthomodular lattices (typical of Quantum logics), and it should be noticed that $b' \rightarrow_D a' = a' + a \cdot b$ is the well-known Sasaki's arrow $a \rightarrow_S b$ that inspired the so-called Q-implications in fuzzy logics, given by $J(x,y) = S(N(x), T(x,y))$. Hence, $[*]$ can be written as $(a \cdot b')' = a \rightarrow_D b$, an identification that seems to be only possible in the framework of Boolean algebras, that is, of classical bivaluated logic, since $\rightarrow_D$ verifies the Modus Ponens inequality $a \cdot (a \rightarrow_D b) \leq b$ whereas $(a \cdot b')'$ does not verify such inequality (in orthomodular lattices).

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