QL-Implications on a finite chain

M. Mas
Dpt. de Matemàtiques i Inf.
Univ. de les Illes Balears
07122 Palma de Mallorca. Spain
dmimmg0@clust.uib.es

M. Monserrat
Dpt. de Matemàtiques i Inf.
Univ. de les Illes Balears
07122 Palma de Mallorca. Spain
dmimma0@clust.uib.es

J. Torrens
Dpt. de Matemàtiques i Inf.
Univ. de les Illes Balears
07122 Palma de Mallorca. Spain
dmijts0@clust.uib.es

Abstract

This paper deals with a kind of implications defined from t-norms, t-conorms and strong negations on a finite chain, through the expression $I(x,y) = S(N(x),T(x,y))$, and called QL-implications. We mainly study those QL-implications derived from smooth t-norms and t-conorms. It is characterized when functions defined in this way are implication functions, and their analytical expression is given. Some additional properties are studied like contrapositive symmetry, the exchange principle and others. In particular, it is proved that a QL-implication satisfies the exchange principle if and only if it is also an S-implication. Finally, some QL-implications are also derived from non smooth t-norms or t-conorms.

Keywords: t-norm, t-conorm, finite chain, smoothness, implication operator, QL-implications.

1 Introduction

In fuzzy logic, implications are generally performed by suitable functions $I : [0,1]^2 \to [0,1]$, called implication operators, derived from t-norms, t-conorms and strong negations. The three most usual ways to define these implication operators are:

i) $I(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}$ for a given left-continuous t-norm T, called *R-implications*,

ii) $I(x,y) = S(N(x),y)$ for a given t-conorm S and a strong negation N, called *S-implications*, and

iii) $I(x,y) = S(N(x),T(x,y))$ for a given t-norm T, a t-conorm S and a strong negation N, called *QL-implications*.

All three cases are extensively studied in the framework of [0,1], (see [2], [3], [4], [11], [12]), characterizing in particular those that satisfy some additional desired properties.

On the other hand, the study of operators defined on a finite chain L is an area of special interest (see [1], [5], [6], [7], [10]), mainly because the expert's reasonings are usually made through a set of linguistic terms or labels which usually is a finite totally ordered set L. This approach is important because numerical interpretations of these labels can be avoided. Frequently, most of the authors working in this line try to translate well known operators on [0,1] (like t-norms and t-conorms) to the case of a finite chain L. Following this idea, a lot of different classes of operators on L are appearing. In particular, smooth t-norms and t-conorms are classified in [10], t-operators and uninorms with a smooth condition on L are characterized in [7] and non-commutative versions can be found in [5] and [8].

Moreover, the first two kinds of implications, R-implications and S-implications, are recently studied in the framework of L (see [9]). However, the third kind of implications, the QL-implications, is not yet studied on L, and the main goal of this paper is to deal with them. It is characterized when QL-functions defined from smooth t-norms and t-conorms, are actually border implications, their analytical expression is given, and some of their properties are studied. It is proved that such a QL-implication satisfies the exchange principle if and only if it is also an S-implication and, this is satisfied only in two cases.

2 Preliminaries

We recall here the smooth t-norms and t-conorms on L and their characterization, that will be used along the paper. From now on, consider the finite chain

$$L = \{0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1\}$$

where $n \geq 1$. Such an L can be understood as a set of linguistic terms or “labels”.

Let us also denote by $[x_i, x_j]$ the finite chain given by the subinterval of all $x_k \in L$ such that $i \leq k \leq j$.

The following two definitions are adapted from [6].

Definition 1 *A function $f : L \to L$ is said to be smooth if it satisfies one of the following conditions:*

- *f is nondecreasing and $f(x_i) = x_j$ implies that $f(x_{i-1}) = x_k$ where k is such that $j-1 \leq k \leq j$.*
- *f is nonincreasing and $f(x_i) = x_j$ implies that $f(x_{i-1}) = x_k$ where k is such that $j \leq k \leq j+1$.*

Definition 2 *A binary operator F on L is said to be smooth if it is smooth in each place.*

Although t-norms, t-conorms and strong negations are usually binary operators on [0,1], they can be defined as in [1] on any partially ordered set and, in particular, on L. In this way, we have the following results:

Proposition 1 *There is only one strong negation on L that is given by*

$$N(x_i) = x_{n+1-i} \quad \textit{for all } x_i \in L. \tag{1}$$

From now on N will always denote the negation on L given by (1).

Proposition 2 (See [10]). *There is one and only one Archimedean smooth t-norm on L given by*

$$T(x_i, x_j) = x_{\max\{0, i+j-(n+1)\}} \tag{2}$$

Moreover, given any subset J of L containing $0, 1$, there is one and only one smooth t-norm on L that has J as the set of idempotent elements. In fact, if J is the set

$$J = \{0 = x_{i_0} < x_{i_1} < \ldots < x_{i_{m-1}} < x_{i_m} = 1\}$$

such a t-norm is given by:

$$T(x_i, x_j) = x_{\max\{i_k, i+j-i_{k+1}\}} \tag{3}$$

if there is an idempotent $x_{i_k} \in J$ such that $x_{i_k} \leq x_i, x_j \leq x_{i_{k+1}}$; and is given by

$$T(x_i, x_j) = \min\{x_i, x_j\}$$

otherwise.

Smooth t-conorms have a classification theorem like the above one for t-norms which can be easily deduced by N-duality. The expression of the only Archimedean t-conorm on L is given by

$$S(x_i, x_j) = x_{\min\{n+1, i+j\}} \tag{4}$$

for all $x_i, x_j \in L$. The following result follows immediately from the previous proposition.

Proposition 3 (See [10]). *There are exactly 2^n different smooth t-norms on L.*

Definition 3 *A binary operator $I : L \times L \to L$ is said to be an implication operator, or an implication, if it satisfies:*

- *I is nonincreasing in the first place and nondecreasing in the second one.*
- *The restriction of I to $\{0,1\}^2$ coincides with the classical implication. That is:*

$$I(0,0) = I(1,1) = 1 \quad \textit{and} \quad I(1,0) = 0.$$

From the definition it follows that $I(x_i, 1) = 1$ for all $x_i \in L$ whereas the symmetrical values $I(1, x_i)$ are not determined in general.

Definition 4 *We will say that an implication $I : L \times L \to L$ is a border implication if it satisfies $I(1, x_i) = x_i$ for all $x_i \in L$.*

3 QL-Implications

Let us begin this section by introducing the operators that will be studied in the paper.

Definition 5 *A binary operator $I : L^2 \to L$ is called a QL-operator when there are a t-norm T and a t-conorm S on L such that I is given by*

$$I(x_i, x_j) = S(N(x_i), T(x_i, x_j)) \tag{5}$$

for all $x_i, x_j \in L$.

Definition 6 *A binary operator $I : L^2 \to L$ is called a QL-implication when it is both a QL-operator and an implication.*

Note that any QL-operator I always satisfies $I(1,x_i) = x_i$ for all $x_i \in L$ and consequently, any QL-implication is in fact a border implication.

The following proposition characterizes the QL-operators obtained from smooth t-norms and smooth t-conorms that are actually QL-implications. Note that in the case of [0,1] only some necessary and sufficient conditions were proved in [12], but no characterization was shown.

Proposition 4 *Let T be a smooth t-norm, S a smooth t-conorm, and I the QL-operator given by (5). Then $I : L^2 \to L$ is a QL-implication if and only if S is the Archimedean t-conorm given by (4).*

From this proposition, a QL-implication given by expression (5) is determined by the t-norm T and thus, it will be denoted by I^T from now on. The general expression of these I^T is stated in the next proposition.

Proposition 5 *Let T_J be the only smooth t-norm with set of idempotents*

$$J = \{0 = x_{i_0} < x_{i_1} < \ldots < x_{i_{m-1}} < x_{i_m} = 1\}.$$

Then the QL-implication I^{T_J} is given by

$I^{T_J}(x_i,x_j) =$

$$= \begin{cases} x_{\max(n+1+i_k-i, n+1+j-i_{k+1})} & \text{if } i,j \in [i_k, i_{k+1}] \\ x_{n+1-i+\min(i,j)} & \text{otherwise.} \end{cases} \quad (6)$$

Let us point out also that the smoothness condition is preserved by QL-implications.

Proposition 6 *Let T be a smooth t-norm, then the corresponding QL-implication I^T is smooth also.*

Remark 1 *It is clear from its definition that any QL-operator I satisfies the following property:*

$$I(x_i,0) = N(x_i) \quad \text{for all } x_i \in L.$$

Unfortunately, other important properties for implications, like the exchange principle

$$I(x_i, I(x_j,x_k)) = I(x_j, I(x_i,x_k))$$

for all $x_i,x_j,x_k \in L$, and contrapositive symmetry

$$I(x_i,x_j) = I(N(x_j),N(x_i))$$

for all $x_i,x_j \in L$, are not satisfied in general for QL-implications. In this way, we have:

Theorem 1 *Let T be a smooth t-norm and I^T the corresponding QL-implication. If I^T satisfies the exchange principle then it also satisfies contrapositive symmetry. Moreover, the following statements are equivalent:*

i) I^T satisfies the exchange principle.

ii) $T = \min$ or T is the Archimedean t-norm.

iii) There is some smooth t-norm T' such that

$$I^T(x_i,x_j) = N(T'(x_i,N(x_j)))$$

for all $x_i,x_j \in L$. That is, I^T is an S-implication where S is the t-conorm N-dual of T'.

Note that,

- when $T = \min$, the corresponding t-norm T' in Theorem 1 is the Archimedean one, obtaining then the Łukasiewicz implication:

$$I^T(x_i,x_j) = x_{\min\{n+1, n+1+j-i\}}$$

- when T is the Archimedean t-norm, the corresponding t-norm T' in Theorem 1 is min, obtaining then the Kleene-Dienes implication:

$$I^T(x_i,x_j) = \max\{x_{n+1-i}, x_j\}$$

4 Some other properties

In this section we deal with some other properties that are many times required for implications, depending on the context. For instance, since any QL-implication is in fact a border implication it is easy to show that

$$I^T(x_i,x_j) \geq x_j \qquad \text{for all} \quad x_i,x_j \in L.$$

On the contrary, the QL-implications I^T satisfy some other properties only for some special t-norms T. Namely:

Proposition 7 *Let T be a smooth t-norm and I^T the corresponding QL-implication. The following statements are equivalent:*

i) $I^T(x_i, x_i) = 1$ *for all* $x_i \in L$.

ii) $I^T(x_i, x_j) = 1$ *if and only if* $x_i \leq x_j$.

iii) T *is the t-norm* min*, i.e.* I^T *is the Łukasiewicz implication.*

Proposition 8 *Let T be a smooth t-norm and I^T the corresponding QL-implication. The following statements are equivalent:*

i) I^T *satisfies the "generalized modus ponens",*

$$T(x_i, I^T(x_i, x_j)) \leq x_j \qquad \text{for all} \quad x_i, x_j \in L.$$

ii) $I^T(x_i, N(x_i)) = N(x_i)$ *for all* $x_i \in L$.

iii) T *is the Archimedean t-norm, i.e.* I^T *is the Kleene-Dienes implication.*

Note that we have limited our study to smooth t-norms and t-conorms but, from non smooth ones, we can also derive QL-implications:

Example 1 *Let S be the nilpotent maximum and T the minimum. Then the operator given by*

$$I(x_i, x_j) = S(N(x_i), T(x_i, x_j))$$

for all $x_i, x_j \in L$ is a QL-implication. Moreover, this implication I is the well known R_0-implication:

$$I(x_i, x_j) = R_0(x_i, x_j) = \begin{cases} x_{n+1} & \text{if } x_i \leq x_j \\ \max(x_{n+1-i}, x_j) & \text{otherwise.} \end{cases}$$

Note that the R_0-implication given in the example above was extensively studied in the framework of [0,1] in [11].

Proposition 9 *Let T be any t-norm and S any t-conorm. If the corresponding QL-operator I is a QL-implication then S satisfies:*

$$S(x_i, N(x_i)) = 1 \qquad \text{for all} \quad x_i, x_j \in L. \qquad (7)$$

However, in this case condition (7) is not sufficient to have a QL-implication. For instance, if T is the drastic t-norm and $n \geq 2$, there is no t-conorm S such that the corresponding QL-operator is a QL-implication.

Acknowledgements

This work has been partially supported by the DGI Spanish project BFM2000-1114.

References

[1] B. De Baets and R. Mesiar (1999), Triangular norms on product lattices, *Fuzzy Sets and Systems*, **104**, pp. 61-75.

[2] H. Bustince, P. Burillo and F. Soria (2003), Automorphisms, negations and implication operators, *Fuzzy Sets and Systems*, **134**, pp. 209-229.

[3] J.C. Fodor (1991), On fuzzy implication operators, *Fuzzy Sets and Systems*, **42**, pp. 293-300.

[4] J.C. Fodor (1995), Contrapositive symmetry on fuzzy implications, *Fuzzy Sets and Systems*, **69**, pp. 141-156.

[5] J.C. Fodor (2000), Smooth associative operations on finite ordinal scales, IEEE Trans. on Fuzzy Systems, **8**, pp. 791-795.

[6] Ll. Godo and C. Sierra (1988), A new approach to connective generation in the framework of expert systems using fuzzy logic, in *Proc. of the XVIIIth ISMVL*, Palma de Mallorca, pp. 157-162.

[7] M. Mas, G. Mayor and J. Torrens (1999), $t-$operators and uninorms on a finite totally ordered set, *Int. J. of Intelligent Systems*, **14** N. 9, pp. 909-922.

[8] M. Mas, M. Monserrat and J. Torrens, On left and right uninorms on a finite chain, accepted in *Fuzzy sets and Systems*.

[9] M. Mas, M. Monserrat and J. Torrens (2002), Operadores de implicación en una cadena finita, in *Proceedings Estylf-2002*. Leon, Spain, pp. 297-302.

[10] G. Mayor and J. Torrens (1993), On a class of operators for expert systems, *Int. J. of Intelligent Systems*, **8** N. 7, pp. 771-778.

[11] D. Pei (2002), R_0 implication: characteristics and applications, *Fuzzy Sets and Systems*, **131**, pp. 297-302.

[12] E. Trillas, C. del Campo and S. Cubillo (2000), When QM-operators are implication functions and conditional fuzzy relations, *Int. J. of Intelligent Systems*, **15** pp. 647-655.