

# On left and right uninorms on a finite chain

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## Abstract

The main concern of this paper is to introduce and characterize the class of operators on a finite chain  $L$ , having the same properties of pseudo-smooth uninorms but without commutativity. Moreover, in this case it will only be required the existence of a one-side neutral element. These operators are characterized as combinations of *AND* and *OR* operators of directed algebras (smooth t-norms and smooth t-conorms) and the case of pseudo-smooth uninorms is retrieved for the commutative case.

**Keywords:** t-norm, t-conorm, uninorm, t-operator, directed algebra, non-commutativity, smoothness, one-side neutral element, finite chain, label.

## 1 Introduction

The study of operators defined on a finite chain  $L$  is an area of increasing interest (see [1], [2], [4], [7], [9]). These operators are related to the management of uncertainty in expert systems. Specifically, the idea is to combine and propagate uncertainty through operators directly defined on the set of linguistic terms or labels which usually is a finite totally ordered set  $L$ . This is important because with this approach, the problems derived from numerical interpretations of these labels disappear.

Frequently, most of the authors which work in this line try to translate well known operators on  $[0,1]$

(like t-norms and t-conorms) to the case of a finite chain  $L$ . Following this idea, a lot of different classes of operators on  $L$  are appearing. In particular, it is given in [9] a detailed study of "directed algebras" which are structures on  $L$  that modelize the structure of continuous De Morgan triplets on  $[0, 1]$ . In the mentioned paper, the idea of continuity is obtained from a condition on the order and, the *AND* and *OR* operators of directed algebra satisfy the same conditions of continuous t-norms and t-conorms on  $[0, 1]$  respectively.

Moreover, t-operators and uninorms (operators on  $[0,1]$  that have been recently characterized, see [3] and [6]) defined on a finite chain  $L$  are also studied in [7]. In this case, the idea of continuity is obtained from the smoothness property. It is proved there, that smooth t-operators as well as pseudo-smooth uninorms on  $L$ , allow to an identical characterization as in the case of  $[0, 1]$ , but now, being the combination between *AND* and *OR* operators of directed algebras.

On the other hand, the non-commutative version of t-operators is also studied for operators defined on  $[0,1]$  and on  $L$ . Really, it is studied the class of operators with the same properties of smooth t-operators but without commutativity. In both cases they are characterized again in a very similar way, see [5] for the case of  $[0,1]$  and [2] for the case of  $L$ .

The main concern of this paper is to introduce and characterize the non-commutative version of uninorms on  $L$  in an analogous way (these operators are already characterized on  $[0,1]$  in [8]). Namely, operators on  $L$  having the same properties of pseudo-smooth uninorms but forgetting

again commutativity. Moreover, in this case it will only be required the existence of a one-side neutral element.

A characterization of these operators is obtained as special combinations of *AND* and *OR* operators of directed algebras (or equivalently, smooth t-norms and smooth t-conorms) and two corollaries are deduced for the commutative case (retrieving the characterization of pseudo-smooth uni-norms) and for the idempotent case (obtaining combinations of minimum and maximum). Thus, new operators on a set of labels are obtained that can also be used for propagation of uncertainty.

## 2 Preliminaries

We recall here different classes of operators on  $L$  and their characterization, that will be used along the paper. From now on, let  $L$  be a finite chain denoted by

$$L = \{0 = x_0 < x_1 < \dots < x_n < x_{n+1} = 1\}$$

Let also denote by  $L_{x_i, x_j}$  the finite chain given by the subinterval  $[x_i, x_j]$  of all  $x_k \in L$  such that  $i \leq k \leq j$ .

**Definition 1** A 5-tuple  $\mathbf{L}=(L, \leq, T, S, N)$  is said to be a directed algebra if it satisfies:

- $T$  and  $S$  are associative, commutative binary operators on  $L$ , with  $T(1,1) = 1$  and  $S(0,0) = 0$ .
- $N : L \rightarrow L$  is an involution such that  $NT(x_i, x_j) = S(Nx_i, Nx_j)$ .
- $x_i \leq x_j$   
     iff there exists  $x_k \in L$  such that  $x_i = T(x_j, x_k)$   
     iff there exists  $x_l \in L$  such that  $x_j = S(x_i, x_l)$ .

In this case  $T$  will be called the *AND* operator and  $S$  the *OR* operator of the directed algebra  $\mathbf{L}$ . We will also say that  $N$  is the negation on  $\mathbf{L}$  and that  $S$  and  $T$  are  $N$ -dual operators.

**Definition 2** Let  $F$  be a binary operator on  $L$ . It is said that  $F$  satisfies the smoothness condition

if whenever  $F(x_i, x_j) = x_k$  then

$$F(x_{i-1}, x_j) = x_l \quad \text{and} \quad F(x_i, x_{j-1}) = x_m$$

where  $l, m$  are such that  $k - 1 \leq l, m \leq k$ .

**Remark:** Note from the definition that a binary operator satisfying the smoothness condition must be non decreasing in each place.

**Proposition 1** (See [7]) Let  $T$  ( $S$ ) be a binary operator on  $L$ . Then,  $T$  ( $S$ ) is an *AND* (*OR*) operator of directed algebra if, and only if, it is associative, commutative, smooth and satisfies  $T(x_i, 0) = 0$  ( $S(x_i, 1) = 1$ ) for all  $x_i \in L$ .

In view of the proposition above, *AND* and *OR* operators of directed algebras will be called from now on, *smooth t-norms* and *smooth t-conorms* respectively.

**Proposition 2** (See [9]). There exists one and only one Archimedean smooth t-norm on  $L$  given by

$$T(x_i, x_j) = x_k \quad \text{where} \quad k = \max(0, i+j-(n+1))$$

Moreover, given any subset  $J$  of  $L$  with  $0, 1 \in J$ , there exists one and only one smooth t-norm on  $L$  whose idempotent elements are exactly those of  $J$ . This smooth t-norm is given by the only Archimedean structure on each subinterval between two consecutive idempotents of  $J$  and by minimum otherwise.

Note that the only possible negation on a directed algebra  $\mathbf{L}$  is given by

$$N(x_i) = x_{n+1-i} \quad \text{for all } x_i \in L$$

and consequently, the structure of smooth t-conorms follows by duality with respect to this negation  $N$ .

On the other hand, another kind of operators were introduced in [2] and characterized as follows:

**Theorem 1** Let  $M : L \times L \rightarrow L$  be a function with  $M(1,0) = x_\lambda, M(0,1) = x_\mu$ . Then  $M$  is associative, non decreasing in each place, smooth and such that  $M(0,0) = 0$  and  $M(1,1) = 1$  if, and only if, one of the following cases follows:

- If  $x_\lambda \leq x_\mu$ , there exist a smooth t-conorm  $S$  on  $L_{0,x_\lambda}$  and a smooth t-norm  $T$  on  $L_{x_\mu,1}$  such that

$$M(x_i, x_j) = \begin{cases} S(x_i, x_j) & \text{if } (x_i, x_j) \in L_{0,x_\lambda}^2 \\ T(x_i, x_j) & \text{if } (x_i, x_j) \in L_{x_\mu,1}^2 \\ x_\lambda & \text{if } x_j \leq x_\lambda \leq x_i \\ x_\mu & \text{if } x_i \leq x_\mu \leq x_j \\ x_j & \text{otherwise} \end{cases}$$

- If  $x_\mu \leq x_\lambda$ , there exist a smooth t-conorm  $S$  on  $L_{0,x_\mu}$  and a smooth t-norm  $T$  on  $L_{x_\lambda,1}$  such that

$$M(x_i, x_j) = \begin{cases} S(x_i, x_j) & \text{if } (x_i, x_j) \in L_{0,x_\mu}^2 \\ T(x_i, x_j) & \text{if } (x_i, x_j) \in L_{x_\lambda,1}^2 \\ x_\lambda & \text{if } x_j \leq x_\lambda \leq x_i \\ x_\mu & \text{if } x_i \leq x_\mu \leq x_j \\ x_i & \text{otherwise} \end{cases}$$

Note that t-operators on  $L$ , introduced in [7], and their characterization follow from the theorem above by adding commutativity.

**Definition 3** A uninorm on  $L$  is a mapping  $U : L \times L \rightarrow L$  which is commutative, associative, increasing in each place and such that there exists an element  $x_e \in L$  (called neutral element) satisfying  $U(x_i, x_e) = x_i$  for all  $x_i \in L$ .

**Proposition 3** (See [7]) If  $U$  is a uninorm on  $L$  satisfying the smoothness condition, then  $U$  is a smooth t-norm or a smooth t-conorm.

In view of the previous result the following definitions were introduced in [7]:

**Definition 4** Let  $U$  be a binary operator on  $L$ . It is said that  $U$  is down-smooth at point  $(x_i, x_j)$  (d.-s. in short) if whenever  $U(x_i, x_j) = x_k$  then

$$U(x_{i-1}, x_j) = x_l \quad \text{and} \quad U(x_i, x_{j-1}) = x_m$$

where  $l, m$  are subject to the condition:  $k - 1 \leq l, m \leq k$ .

**Remark:** It is clear from this definition that any binary operator  $U$  on  $L$  is d.-s. at point  $(0, 0)$ . Moreover,  $U$  satisfies the smoothness condition if and only if it is d.-s. at any point  $(x_i, x_j) \in L^2$ . Note also that a similar definition of up-smoothness could be given alternately with similar properties.

Now, let us introduce the idea of “pseudo-smoothness” through the following definitions.

**Definition 5** Let  $U$  be a uninorm on  $L$  with neutral element  $x_e$ .

- It is said that  $U$  is left d-pseudosmooth if it satisfies d.-s. at any point in  $L^2$  except perhaps at points  $(x_i, x_e), (x_e, x_i)$  with  $x_i \in \{x_{e+1}, \dots, 1\}$ .
- It is said that  $U$  is right d-pseudosmooth if it satisfies d.-s. at any point in  $L^2$  except perhaps at points  $(x_i, x_{e+1}), (x_{e+1}, x_i)$  with  $x_i \in \{0, \dots, x_{e-1}\}$ .

Next theorem gives us the structure of the left and right d-pseudosmooth uninorms on  $L$ .

**Theorem 2** (See [7]). Let  $U$  be a uninorm on  $L$  with neutral element  $0 < x_e < 1$ .  $U$  satisfies left d-pseudosmoothness if, and only if,  $U(0, 1) = 0$  and  $U$  is given by:

$$U(x_i, x_j) = \begin{cases} T(x_i, x_j) & \text{if } (x_i, x_j) \in L_{0,x_e}^2 \\ S(x_i, x_j) & \text{if } (x_i, x_j) \in L_{x_e,1}^2 \\ \min(x_i, x_j) & \text{otherwise} \end{cases}$$

where  $T$  is a smooth t-norm on  $L_{0,x_e}$  and  $S$  is a smooth t-conorm on  $L_{x_e,1}$ .

A similar characterization for right d-pseudosmooth uninorms works replacing  $\min$  by  $\max$ .

### 3 Left and right uninorms

**Definition 6** An operator  $U : L^2 \rightarrow L$  is said to be a left-uninorm if it is nondecreasing, associative and such that there exists a left neutral element, i.e. an element  $x_e \in L$  satisfying  $U(x_e, x_j) = x_j$  for all  $x_j \in L$ .

**Definition 7** An operator  $U : L^2 \rightarrow L$  is said to be a right-uninorm if it is nondecreasing, associative and such that there exists a right neutral element, i.e. an element  $x_e \in L$  satisfying  $U(x_i, x_e) = x_i$  for all  $x_i \in L$ .

**Remark:** It follows immediately from any of the two definitions above that  $U(0, 0) = 0$  and  $U(1, 1) = 1$ . Note also that the one-side neutral

element needs not to be unique: The projection operator given by  $U(x_i, x_j) = x_j$  for all  $x_i, x_j \in L$  is such that any element in  $L$  is a left neutral element.

**Definition 8** Let  $U : L^2 \rightarrow L$  be a left (or right) uninorm. Then  $U$  is said to be left d-pseudosmooth, if there exists a left (respectively a right) neutral element  $x_e \in L$  of  $U$  such that the set of points in which  $U$  is not d.-s. is contained in

$$\{(x_i, x_e) \in L^2 \mid i > e\} \cup \{(x_e, x_j) \in L^2 \mid j > e\}.$$

**Definition 9** Let  $U : L^2 \rightarrow L$  be a left (or right) uninorm. Then  $U$  is said to be right d-pseudosmooth, if there exists a left (respectively a right) neutral element  $e \in L$  of  $U$  such that the set of points in which  $U$  is not d.-s. is contained in

$$\{(x_i, x_{e+1}) \in L^2 \mid i < e\} \cup \{(x_{e+1}, x_j) \in L^2 \mid j < e\}.$$

The main concern of this paper is to characterize left d-pseudosmooth left uninorms, but a similar characterization can be given for right d-pseudosmooth left uninorms and also for the cases of right uninorms.

**Proposition 4** Let  $U : L^2 \rightarrow L$  be a left uninorm and let  $0$  be a left neutral element for which  $U$  is left d-pseudosmooth. Then

$$U(x_i, x_j) = \begin{cases} S(x_i, x_j) & \text{if } (x_i, x_j) \in L_{0, x_\beta}^2 \\ x_\beta & \text{if } x_j \leq x_\beta \leq x_i \\ x_j & \text{otherwise} \end{cases} \quad (1)$$

where  $x_\beta = U(1, 0)$  and  $S$  is a smooth t-conorm on  $L_{0, x_\beta}$ .

In the following lemmas and proposition, we will suppose that  $U$  has a left neutral element  $x_e > 0$ . Also, we will denote  $U(0, 1) = x_\alpha$  and  $U(1, x_e) = x_\beta$ .

**Lemma 1** Let  $U : L^2 \rightarrow L$  be a left uninorm. Then  $x_\alpha = 1$  or  $x_\alpha < x_e$ , and also  $x_\beta \geq x_e$ .

From now on, given any left d-pseudosmooth left uninorm  $U$ ,  $x_e$  will denote a left neutral element for which  $U$  is left d-pseudosmooth.

**Lemma 2** Let  $U : L^2 \rightarrow L$  be a left d-pseudosmooth left uninorm. Then  $U(1, 0) = 0$ ,  $U(x_i, x_j) = x_j$  for all  $(x_i, x_j) \in [x_e, 1] \times [0, x_e]$  and  $U(x_i, x_j) = x_j$  for all  $(x_i, x_j) \in [0, x_e] \times [0, x_\alpha]$ .

**Proposition 5** Let  $U : L^2 \rightarrow L$  be a left d-pseudosmooth left uninorm. If  $x_\alpha = 1$ , then  $U(x_i, x_j) = x_j$  for all  $x_i, x_j \in L$ . (Note that this is a degenerated case already contained in the case  $x_e = 0$ ).

Let us take  $x_\alpha < x_e$  in the following lemmas.

**Lemma 3** Let  $U : L^2 \rightarrow L$  be a left uninorm. Then  $U(x_i, x_j) = x_\alpha$  for all  $(x_i, x_j) \in [0, x_\alpha] \times [x_\alpha, 1]$ .

**Lemma 4** Let  $U : L^2 \rightarrow L$  be a left d-pseudosmooth left uninorm. Then  $U(x_i, x_e) = x_i$  for all  $x_i \in [x_\alpha, x_e]$  and  $U(x_i, x_j) = x_i$  for all  $(x_i, x_j) \in [x_\alpha, x_e] \times [x_e, 1]$ .

**Lemma 5** Let  $U : L^2 \rightarrow L$  be a left d-pseudosmooth left uninorm. Then in  $[x_e, 1]^2$   $U$  has the form given by proposition 4 taking  $L_{x_e, 1}$  as the finite chain and, restricted to  $[x_\alpha, x_e]^2$ ,  $U$  is a smooth t-norm on the chain  $L_{x_\alpha, x_e}$ .

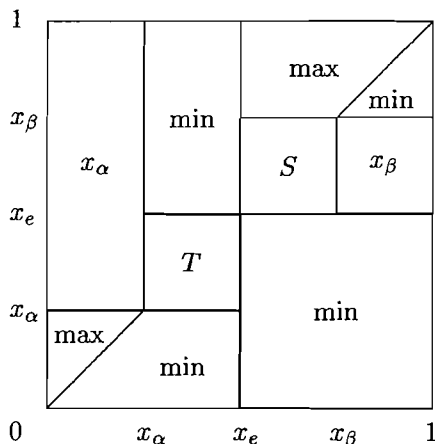
**Theorem 3** Let  $U : L^2 \rightarrow L$  be any binary operator. Then  $U$  is a left d-pseudosmooth left uninorm if, and only if, there exist  $x_e \in L$ , a smooth t-norm on  $L_{x_\alpha, x_e}$   $T$  and a smooth t-conorm  $S$  on  $L_{x_e, x_\beta}$  such that one of the following cases hold, where  $x_\alpha = U(0, 1)$  and  $x_\beta = U(1, x_e)$ ,

- $x_e = 0$  and then  $x_\alpha = 1$ , and  $U$  is given by expression (1).
- $x_e > 0$  and then  $x_\alpha < x_e \leq x_\beta$ , and  $U$  is given by

$$U(x_i, x_j) = \begin{cases} x_\alpha & \text{if } x_i \leq x_\alpha \leq x_j \\ T(x_i, x_j) & \text{if } x_\alpha \leq x_i, x_j \leq x_e \\ S(x_i, x_j) & \text{if } x_e \leq x_i, x_j < x_\beta \\ x_\beta & \text{if } x_e \leq x_j \leq x_\beta \leq x_i \\ x_i & \text{if } x_\alpha \leq x_i < x_e \leq x_j \\ x_j & \text{otherwise} \end{cases} \quad (2)$$

or  $x_\alpha = 1, x_\beta = e$  and  $U(x_i, x_j) = x_j$  for all  $x_i, x_j \in L$ .

The general structure of left d-pseudosmooth left uninorms is presented in the following figure:



**Corollary 1** Let  $U$  be a left  $d$ -pseudosmooth left uninorm with  $x_e > 0$ . The following assertions are equivalent

- i)  $U$  is a commutative operator
- ii)  $U(0,1) = 0$  and  $U(1, x_e) = 1$ .
- iii)  $U$  is a left  $d$ -pseudosmooth uninorm.

**Corollary 2** Let  $U$  be a left  $d$ -pseudosmooth left uninorm. Then  $U$  is idempotent if, and only if, there exists  $x_e \in L$  such that  $U$  is given by expression (1) or (2) replacing  $T$  by  $\min$  and  $S$  by  $\max$ .

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