Strict Triangular Norms and Characterization of T-Measures

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Abstract

We characterize T-measures on weakly generated tribes, where T is a strict triangular norm and we give a Liapunoff Theorem for these measures. This generalizes previous results obtained for monotonic T-measures or for Frank triangular norms.

Keywords: Triangular norm, T-tribe, T-measure, Frank triangular norm, nearly Frank triangular norm, Liapunoff Theorem.

1 The notion of T-measure

The classical measure and probability theory is based on the notion of σ-algebra of subsets of a set. Butnariu and Klement [6] generalized it to fuzzy sets by considering their collections called *T-tribes*. Here T denotes a fixed triangular norm (serving as a fuzzy conjunction), i.e., a binary operation $T: [0,1]^2 \to [0,1]$ which is commutative, associative, nondecreasing, and satisfies the boundary condition $T(a,1) = a$ for all $a \in [0,1]$ (see [8, 13]). A *T-tribe* is a collection of fuzzy subsets which contains the empty set and which is closed with respect to the standard fuzzy complement and (the pointwise application of) the triangular norm T (extended to countably many arguments). Here we do not need this notion in its full generality; we restrict our attention to weakly generated tribes. (In fact, also Butnariu and Klement studied only weakly generated tribes; moreover, they considered only Frank triangular norms for T.)

Let X be a nonempty set. A collection $\mathcal{T} \subseteq [0,1]^X$ is called a *weakly generated tribe* [11] if there is a σ-algebra $\mathcal{B}$ of subsets of X and a σ-ideal Δ in $\mathcal{B}$ such that $\mathcal{T}$ is the collection of all functions $A: X \to [0,1]$ satisfying the following two conditions:

- A is $\mathcal{B}$-measurable,
- $A^{-1}((0,1)) \in \Delta$.

If, moreover, $\Delta = \{\emptyset\}$, then we call $\mathcal{T}$ a *generated tribe* [5, 6].

Let T be a triangular norm and $S: [0,1] \to [0,1]$ its dual triangular conorm, i.e.,

$$S(a,b) = 1 - T(1-a, 1-b).$$

A function $m: \mathcal{T} \to \mathbf{R}$ is a *T-measure* if it satisfies the following axioms:

$$m(0) = 0,$$

$$m(T(A,B)) + m(S(A,B)) = m(A) + m(B),$$

$$A_n \nearrow A \Longrightarrow m(A_n) \to m(A),$$

where the symbol $\nearrow$ denotes monotone increasing convergence.

The notion of T-measure is not only a natural generalization of a classical measure. It is also the base of successful applications in game theory. Many deep mathematical results, including a generalization of Liapunoff Theorem, were proved in [1, 2, 6].

2 Nearly Frank triangular norms

To formulate our results, we need a few more notions.

A continuous triangular norm T is called *strict* if

$$a < b,\ 0 < c \Longrightarrow T(a,c) < T(b,c).$$

The *Frank family of triangular norms* T_s, $s \in [0,\infty]$, was defined in [7]. For $s \in (0,\infty)\setminus\{1\}$, the Frank triangular norms are defined by the formula

$$T_s(a,b) = \log_s \left(1 + \frac{(s^a-1)(s^b-1)}{s-1}\right).$$

For $s = 1$ it is defined by

$$T_1(a,b) = a \cdot b$$

(the product triangular norm). These are all strict Frank triangular norms; we do not deal here with Frank triangular norms which are not strict.

By an (order) *automorphism* of $[0,1]$ we mean an increasing bijection.

Definition 2.1 : *An automorphism h of $[0,1]$ is called a* negation-preserving automorphism *if it commutes with the standard fuzzy negation, i.e.,*

$$h(1-a) = 1 - h(a)$$

for all $a \in [0,1]$.

Definition 2.2 : *A triangular norm T is called* nearly Frank *if there is a Frank triangular norm T_s and a negation-preserving automorphism h satisfying the equation*

$$T(a,b) = h^{-1}(T_s(h(a),h(b))).$$

Strict nearly Frank triangular norms correspond to strict Frank triangular norms.

3 Results

A characterization of monotonic T-measures for a Frank triangular norm T has been presented in [6] and completed in [10]. Later on, we characterized monotone T-measures on generated tribes for any non-Frank strict triangular norm T [12]. Independently, a generalization for nonmonotonic T-measures and Frank triangular norms was done in [3]. Here we present a common generalization [4]: We characterize nonmonotonic T-measures on weakly generated tribes, where T is a strict triangular norm. Our main new contribution may be stated as follows:

Theorem 3.1 *Let T be a strict triangular norm which is not nearly Frank and let $\mathcal{T}$ be a weakly generated tribe. Then each T-measure μ on $\mathcal{T}$ is of the form*

$$m(A) = \mu(A^{-1}((0,1])),$$

where μ is a (classical) measure on $\mathcal{B}$.

A consequence of [3] and [12] yields

Theorem 3.2 *Let T be a strict triangular norm which is nearly Frank and let $\mathcal{T}$ be a weakly generated tribe. Then each T-measure μ on $\mathcal{T}$ is of the form*

$$\mu(A) = \nu(A^{-1}((0,1])) + \int h \circ A \,\mathrm{d}\lambda,$$

where ν, λ are (classical) measures on $\mathcal{B}$ and h is the negation-preserving automorphism from Definition 2.2.

These results complete the previous investigations and also show the specific role of Frank [7] and nearly Frank [9, 12] triangular norms—only these triangular norms admit T-measures which are not of the form from Theorem 3.1. Moreover, they allow to generalize Liapunoff Theorem about the range to the context of measures based on strict triangular norms.

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