

The Group of Isometries of an Indistinguishability Operator

J. Jacas

Sec. Matemàtiques i Informàtica - ETSAB
Univ. Politècnica de Catalunya
Avda. Diagonal 649
08028 Barcelona (Spain)
jacas@ea.upc.es

J. Recasens

Sec. Matemàtiques i Informàtica - ETSAV
Univ. Politècnica de Catalunya
Pere Serra 1-15
08190 Sant Cugat del Vallès. Barcelona. (Spain)
recasens@ea.upc.es

Abstract

This paper ¹ studies some geometric aspects of indistinguishability operators (also called similarities and fuzzy equivalences). Concretely, it will be focused on the (geometric) group associated to a T -indistinguishability operator E on X (i.e., the group of all bijective maps $h : X \rightarrow X$ such that $E(x, y) = E(h(x), h(y)) \quad \forall x, y \in X$).

The cases for E being one-dimensional and invariant under translations on the real line will be completely studied.

Keywords: indistinguishability operator, similarity, fuzzy equivalence relation, fuzzy equality, isometry.

1 Introduction

Indistinguishability operators generalize the crisp equivalence relations to the fuzzy framework. They have been widely studied and they have proved to be useful in different contexts such as fuzzy control, approximate reasoning or cluster analysis. Depending on the authors and on the t-norms used, they have received other names such as fuzzy equivalences, fuzzy equalities or similarity relations, among others (see [1] for more information).

In this paper, we study some geometric aspects of these operators by using some ideas contained in

¹Research supported by DGICYT project number PB98-0924

[7]. Concretely, it will be focused on the (geometric) group associated to a T -indistinguishability operator E on X (i.e., the group of all bijective maps $h : X \rightarrow X$ such that $E(x, y) = E(h(x), h(y)) \quad \forall x, y \in X$).

The case for E being one-dimensional and invariant under translations on the real line will be completely studied. The fact that the natural indistinguishability operators [1] E_T on $[0,1]$ are a special kind of one-dimensional ones, make their study especially interesting. The indistinguishability operators invariant under translations are widely used in applications (e.g. in fuzzy control) and therefore, it is worthwhile to study and to characterize them.

For the sake of completeness, we recall in the next section some known results on indistinguishability operators.

2 Preliminaries

Definition 2.1. Given a t -norm T , a T -indistinguishability operator E on a set X is a reflexive and symmetric fuzzy relation on X such that $T(E(x, y), E(y, z)) \leq E(x, z)$ (T -transitivity), for all $x, y, z \in X$

E separates points iff $E(x, y) = 1$ implies $x = y$.

Definition 2.2. Given a continuous t -norm T , its quasi-inverse or residuation T is defined $\forall x, y \in [0, 1]$ by

$$\hat{T}(x|y) = x \rightarrow_T y = \sup\{\alpha \in [0, 1] | T(\alpha, x) \leq y\}.$$

Lemma 2.3. [8] Every fuzzy set μ of X generates

a T -indistinguishability operator E_μ on X in the following way:

$$E_\mu(x, y) = \hat{T}(\text{Max}(\mu(x), \mu(y)) | \text{Min}(\mu(x), \mu(y))) \\ = \text{Min}(\mu(x) \rightarrow_T \mu(y), \mu(y) \rightarrow_T \mu(x))$$

Trivially, E_μ separates points if and only if μ is an injective map.

Example 2.4. If L is the Lukasiewicz t -norm, i.e. $L(x, y) = \text{Max}(x + y - 1, 0)$, then $E_\mu(x, y) = 1 - |\mu(x) - \mu(y)|$.

If Π is the Product t -norm $\Pi(x, y) = x \cdot y$ then

$$E_\mu(x, y) = \text{Min}\left(\frac{\mu(x)}{\mu(y)}, \frac{\mu(y)}{\mu(x)}\right).$$

If T is the Minimum t -norm $\text{Min}(x, y)$, then

$$E_\mu(x, y) = \begin{cases} 1 & \text{if } \mu(x) = \mu(y) \\ \text{Min}(\mu(x), \mu(y)) & \text{otherwise.} \end{cases}$$

Definition 2.5. Given a continuous t -norm T , the *natural* T -indistinguishability operator on $[0, 1]$ is defined by

$$E_T(x, y) = \hat{T}(\text{Max}(x, y) | \text{Min}(x, y)) = \\ \text{Min}(x \rightarrow_T y, y \rightarrow_T x).$$

In this case,

$$E_L(x, y) = 1 - |x - y| \\ E_\Pi(x, y) = \text{Min}\left(\frac{x}{y}, \frac{y}{x}\right) \\ E_{\text{Min}}(x, y) = \begin{cases} 1 & \text{if } x = y \\ \text{Min}(x, y) & \text{otherwise.} \end{cases}$$

3 Isometries between Indistinguishability operators

Definition 3.1. Given two sets X, Y and two T -indistinguishability operators E, F on X, Y respectively, a map $\tau : X \rightarrow Y$ is an isometry if and only if $E(x, y) = F(\tau(x), \tau(y)) \forall x, y \in X$.

Lemma 3.2. With the previous notations, given an isometry $\tau : X \rightarrow Y$, if F separates points and τ is injective, then E also separates points.

Before studying the isometries between indistinguishability operators, it is convenient to analyze when two fuzzy sets μ, ν generate the same operator (i.e. $E_\mu = E_\nu$).

Theorem 3.3. Let T be an archimedean t -norm, t a generator of T [6] and μ, ν fuzzy subsets of X , $E_\mu = E_\nu$ if and only if, $\forall x \in X$ one of the following conditions holds:

- a) $t(\mu(x)) = t(\nu(x)) + k_1$ with $k_1 \geq \sup\{-t(\nu(x)) | x \in X\}$,
- b) $t(\mu(x)) = -t(\nu(x)) + k_2$ with $k_2 \geq \sup\{t(\nu(x)) | x \in X\}$.

Moreover, if T is non strict, then $k_1 \leq \inf\{t(0) - t(\nu(x)) | x \in X\}$ and $k_2 \leq \inf\{t(0) + t(\nu(x)) | x \in X\}$.

Example 3.4. If T is the Lukasiewicz t -norm L , we can take $t(x) = 1 - x$ and with the previous notations $\mu(x) = \nu(x) + k$ with $\inf\{1 - \nu(x)\} \geq k \geq \sup\{-\nu(x)\}$ or $\mu(x) = -\nu(x) + k$ with $\inf\{1 + \nu(x)\} \geq k \geq \sup\{\nu(x)\}$.

Example 3.5. If T is the product, we can take $t(x) = -\log x$ and then $\mu(x) = \frac{\nu(x)}{k}$ with $k \geq \sup\{\nu(x)\}$ or $\mu(x) = \frac{k}{\nu(x)}$ with $k \leq \inf\{\nu(x)\}$.

Theorem 3.6. Let T be the t -norm Minimum and let μ be a fuzzy subset of X such that there exists an element x_M of X with $\mu(x_M) \geq \mu(x) \forall x \in X$. Let $Y \subset X$ be the set of elements x of X with $\mu(x) = \mu(x_M)$ and $s = \sup\{\mu(x) | x \in X - Y\}$. A fuzzy subset ν of X generates the same T -indistinguishability operator than μ if and only if $\forall x \in X - Y$,

$$\mu(x) = \nu(x) \text{ and } \nu(y) = t \text{ with } s \leq t \leq 1 \quad \forall y \in Y.$$

Next theorem relates the previous results with the group of isometries of an indistinguishability operator.

Theorem 3.7. Let T be a continuous t -norm and E_μ the T -indistinguishability operator on X generated by a fuzzy subset μ . The map $\tau : X \rightarrow X$ is an isometry if and only if there exists a fuzzy subset ν of X with $E_\mu = E_\nu$ and $\mu \circ \tau = \nu$.

Corollary 3.8. Let T be a continuous t -norm and E_μ, E_ν two T -indistinguishability operators on X, Y respectively generated by μ and ν . The map $\tau : X \rightarrow Y$ is an isometry if and only if $\mu = \nu \circ \tau$.

4 The group of isometries of E_T

The following universal property of the natural indistinguishability operator gives special interest to its study.

Lemma 4.1. Let T be a continuous t -norm, E_μ the T -indistinguishability operator on X generated by μ and E_T the natural T -indistinguishability operator on $[0,1]$. The membership function μ is an isometry between E_μ and E_T .

In fact, the preceding results can be translated to the language of categories in such a way that E_T is a final universal object (see [4]).

Let us calculate the group of isometries of E_T .

Theorem 4.2. Let T be a non-strict archimedean t -norm and t an additive generator of T . The group of isometries of E_T consists of the identity and the strong negation φ generated by t ($\varphi(x) = x \rightarrow_T 0$).

Definition 4.3. Given a t -norm T and $a \in [0, 1]$, the map $t_a : [0, 1] \rightarrow [0, 1]$ defined by $t_a(x) = T(a, x)$ will be called a T -translation.

Theorem 4.4. Let T be a strict archimedean t -norm. The group of isometries of E_T is the set of T -translations of $[0,1]$ (i.e. $\{t_a | a \in [0, 1]\}$).

Theorem 4.5. Let T be the t -norm Minimum. The group of isometries of E_T consists of only the identity map.

As a corollary of these theorems, we can answer an interesting question relating indistinguishability operators and negations. In fuzzy logic E_T can be viewed as the bimplication in the sense that $E_T(x, y)$ expresses the degree of equivalence between x and y . It seems reasonable to demand

that the degree of equivalence between elements coincide with the degree of their negations (i.e. $E_T(x, y) = E_T(\varphi(x), \varphi(y))$). The following question arises therefore naturally: When a negation is an isometry of E_T ?

Corollary 4.7. Given an archimedean t -norm T and a strong negation φ , φ is an isometry of E_T if and only if T is non strict and T and φ have a common generator.

5 T -indistinguishability operators invariant by translations

It is usual in fuzzy control to model the values of the linguistic variables by fuzzy numbers. Fuzzy numbers can be thought of as equivalence classes of a fuzzy equivalence relation (i.e. a T -indistinguishability operator) in the sense that fuzzy numbers are needed because the equality used in the real line is not crisp due to lack of accuracy and vagueness [2]. In many applications, fuzzy numbers of the same shape are used so that they only differ by a translation. For this reason, it is interesting to determine when a T -indistinguishability operator on the real line R is invariant under translations in the following sense:

Definition 5.1. Let E be a T -indistinguishability operator on R . E is *invariant by translations* if and only if $\forall x, y, a \in R$, $E(x, y) = E(x + a, y + a)$.

Therefore, E is invariant by translations if and only if the group of isometries of E admits the group of translations of R as a subgroup.

Definition 5.2. A *fuzzy number* is a map $f_a : R \rightarrow [0, 1]$ such that there exists $a \in R$ with $f_a(a) = 1$ and non decreasing in $(-\infty, a)$ and non increasing in $(a, +\infty)$.

Lemma 5.3. Let E be an invariant by translations T -indistinguishability operator. Then for every $a, c \in R$

$$T(E(0, a), E(0, c)) \leq E(0, a + c).$$

Proof.

$$\begin{aligned} T(E(0, a), E(0, c)) &= T(E(-a, 0), E(0, c)) \\ &\leq E(-a, c) = E(0, a + c). \end{aligned}$$

In particular,

$$T(E(0, a), E(0, na)) \leq E(0, (n + 1)a) \forall n \in N.$$

A T -indistinguishability operator will be called admissible if and only if its singletons or columns [2,3,5,9] are fuzzy numbers.

Theorem 5.4. Let T be the Minimum t -norm. E is a T -indistinguishability operator admissible and invariant under translations if and only if there exists $k \in [0, 1]$ such that

$$E(x, y) = \begin{cases} 1, & \text{if } x = y \\ k. & \text{otherwise.} \end{cases}$$

Proof. \Rightarrow)

Let $a > 0$. From Lemma 5.3.

$$\text{Min}(E(0, a), E(0, na)) \leq E(0, (n + 1)a) \quad \forall n \in N$$

and since E is admissible $\text{Min}(E(0, a), E(0, na)) = E(0, na)$ so that

$$E(0, na) \leq E(0, (n + 1)a).$$

But since E is admissible, $E(0, na) \geq E(0, (n + 1)a)$.

Therefore $E(0, na) = E(0, (n + 1)a) \forall a > 0, \forall n \in N$. Due to the monotonicity of $E(0, \cdot)$ in $(0, +\infty)$, $E(0, \cdot)$ must be constant in this interval.

A similar argument works with $a < 0$ for $(-\infty, 0)$.

Now the results follows applying the invariance under translations of E .

\Leftarrow)

Trivial

Theorem 5.5. Let T be an archimedean t -norm with generator t . E is an admissible and invariant under translations T -indistinguishability operator if and only if there exists a map $F : R \rightarrow R$ which is non decreasing and subadditive in R^+ ,

with $F(0) = 0$ and $F(x) = F(-x) \forall x \in R$, such that $E(x, y) = t^{[-1]} \circ F(y - x)$.

Proof. The inequality of Lemma 3.3 can be written

$$t^{[-1]}(t(E(0, a)) + t(E(0, c))) \leq E(0, a + c)$$

or

$$t(E(0, a)) + t(E(0, c)) \geq t(E(0, a + c)).$$

If F is defined by $F(x) = t(E(0, x))$, then $F(a) + F(b) \geq F(a+b)$ and $F(0) = 0$, $F(x) = F(-x)$ and is non decreasing in $(0, +\infty)$ $E(0, x) = t^{[-1]}(F(x))$ and $E(y, x) = E(0, y - x) = t^{[-1]}(F(y - x))$.

References

- [1] Boixader, D., Jacas, J. and Recasens, J. (2000) Fuzzy Equivalence Relations: Advanced Material. In Dubois, Prade Eds. *Fundamentals of Fuzzy Sets*, Kluwer, 261-290.
- [2] Jacas, J. and Recasens, J. (1993) Fuzzy numbers and equality relations, Proc. FUZZIEEE'93, San Francisco
- [3] Jacas, J. and Recasens, J. (1995) Fuzzy T-transitive relations: eigenvectors and generators. *Fuzzy Sets and Systems*, **72** 147-154.
- [4] Jacas, J. and Recasens, J. (2001) Maps and isometries between indistinguishability operators. *Soft Computing*, (To appear).
- [5] De Soto, A.R., Recasens, J. (2000) Modelling a Linguistic Variable as a Hierarchical Family of Partitions induced by an Indistinguishability Operator. *Fuzzy Sets and Systems*, **121** 427-437.
- [6] Schweizer, B., Sklar, A. (1983) *Probabilistic Metric Spaces*, North-Holland, Amsterdam.
- [7] Trillas, E., Alsina, C. (1978) Introducción a los Espacios Métricos Generalizados. Fund. J. March. Serie Universitaria, **49**, Madrid.
- [8] Valverde, L. (1985) On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems*, **17**, 313-328.
- [9] Zadeh, L. A. (1971) Similarity relations and fuzzy orderings. *Inform. Sci.* **3**, 177-200.