Imprecision of the Elements of Fuzzy Eigenvectors of Pairwise Comparison Matrices in Fuzzy AHP

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Abstract

Fuzzy AHP is often used in multi-criteria decision analysis. In this paper we consider an example given by Deng [1] and in particular the method used to approximate the fuzzy principle eigenvector of the fuzzy pairwise comparison matrices. We show that the imprecision of the fuzzy numbers obtained as outputs is highly correlated to the value of the kernel of the fuzzy number. We also show that output data obtained using a different approximation method exhibits a similar relationship. On the basis of these results we question whether the Fuzzy AHP propagates uncertain inputs effectively to uncertain outputs.

Keywords: Fuzzy AHP, Fuzzy Arithmetic, Fuzzy Eigenvectors.

1 Introduction

In a recent paper Deng [1] advocated the use of the Fuzzy AHP in multi-criteria decision analysis and gave a numeric example of how the technique could be applied. We look again at this example and in particular the method used to approximate the fuzzy principle eigenvector of the pairwise comparison matrices. We show that the imprecision of the fuzzy numbers obtained as elements of the fuzzy eigenvectors is highly correlated to the value of the kernels of those fuzzy numbers.

2 Background - The Analytic Hierarchy Process (AHP)

The analytic hierarchy process (AHP) developed by Saaty [4] is often used in multi-attribute decision analysis where the data is qualitative rather than numeric.

Table 1: Values for Saaty's Analytic Hierarchy Process

Numerical Scale a_{ij}	Relative Importance
1	*i* and *j* are equally important
3	*i* is moderately more important compared to *j*
5	*i* is strongly more important compared to *j*.
7	*i* is very strongly more important compared to *j*
2,4,6,8	are used as intermediate values

Pairwise comparison of decision criteria are used to set up a reciprocal matrix the principle right eigenvector of which gives the relative importance of the various criteria as follows:

1. Select a finite set of $n \in \mathbb{N}$ criteria.
2. Let **A** be an $n \times n$ matrix, with each criteria labeling one row and one column of the matrix.
3. Compare each pair of requirements. Decide if i or j is more important assign a_{ij} a numerical value based on the scale in table 1.

4. If criteria j is more important than requirement i then assign $1/a_{ij}$. Note that $a_{ij} * a_{ji} = 1$ and $a_{ii} = 1$.

5. Find the eigenvector $\mathbf{v}$ corresponding to the maximum eigenvalue $\lambda_{\max}$ of $\mathbf{A}$; normalise $\mathbf{v}$ to get the vector $\mathbf{w}$ of weights w_i.

$$\mathbf{A} = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ 1/a_{21} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{n1} & 1/a_{n2} & \dots & 1 \end{bmatrix} \qquad (1)$$

Two techniques are used to approximate $\mathbf{w}$, for crisp consistent pairwise comparison matrices where $a_{ik} \cdot a_{kj} \doteq a_{ij}\ \forall i,k,j$ both methods give the same results as an eigenvalue calculation. The first is to find the normalised geometric mean of each of the rows of $\mathbf{A}$. The second finds the normalised row sums of $\mathbf{A}$. Fuzzy extensions of the AHP applying these approximations use the standard arithmetic operations on positive triangular fuzzy numbers, because of their computational simplicity. For strictly positive triangular fuzzy numbers $A = \langle a_1, a_2, a_3 \rangle$ and $B = \langle b_1, b_2, b_3 \rangle$ with kernels a_2 and b_2 these are as follows:-

$$A^{-1} = \langle a_3^{-1},\ a_2^{-1},\ a_1^{-1} \rangle;$$
$$A + B = \langle a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3 \rangle;$$
$$A - B = \langle a_1 - b_3,\ a_2 - b_2,\ a_3 + b_1 \rangle;$$
$$\forall k \in \mathbb{R} \geq 0,\ kA = \langle ka_1,\ ka_2,\ ka_3 \rangle,$$
$$\forall k \in \mathbb{R} \leq 0,\ kA = \langle ka_3,\ ka_2,\ ka_1 \rangle;$$
$$AB = \langle a_1 b_1,\ a_2 b_2,\ a_3 b_3 \rangle;$$
$$A/B = \langle a_1/b_3,\ a_2/b_2,\ a_3/b_1 \rangle$$

and

$$\forall p \in R,\ A^p = \langle a_1^p,\ a_2^p,\ a_3^p \rangle.$$

2.1 Imprecision of fuzzy numbers

The imprecision of a fuzzy number is related to the imprecision of its α-cuts which are closed intervals. The total imprecision of a fuzzy number is the combination of the imprecision in every level α. For a triangular fuzzy number this is the area under the membership function suitably scaled to reflect the measurement scale [2].

3 Desirable features for the eigenvalue calculation

We set out below what seem to be desirable features of any methods used to calculate the fuzzy principle eigenvector of a fuzzy pairwise comparison matrix.

1. That the kernel of the fuzzy eigenvector is equal to the eigenvector of the crisp pairwise comparison matrix which always met as the a_2 value of a triangular fuzzy number is the same as the crisp value

2. That the different methods of calculation should produce comparable outputs.

3. That there should be a consistent relationship between the imprecision of the elements of the fuzzy pairwise comparison matrices and the imprecision of the associated fuzzy eigenvectors.

4 Imprecision of inputs and outputs

The inputs to the system are the fuzzy numbers shown in table 1. To enable the imprecision of these fuzzy numbers to be compared with the outputs we first scale them to get $\text{imp}(\tilde{1})/10 = 0.1$ and $\text{imp}(\tilde{i})/10 = 0.2$ for $i \in \mathbb{N},\ 2 \leq i \leq 10$.

Using the fuzzy numbers and matrices from [1] set out in tables 2 and 3 it was found that the imprecision of the outputs varied from a minimum of 0.085 to a maximum of 0.58 with an average of 0.29 using the row sum method (Figure 1) and from a minimum of 0.051 to a maximum of 0.81 with an average of 0.35 using the geometric mean method (Figure 2).

Table 2: Fuzzy Numbers used in [1]

Triangular Fuzzy Number	Membership Function
$\tilde{1}$	$\langle 1,1,3 \rangle$
$\tilde{x}$	$\langle (x-2), x, (x+2) \rangle$ for $x = 3,5,7$
$\tilde{9}$	$\langle 7,9,11 \rangle$

It is therefore apparent that neither method propagates imprecision from the inputs to the outputs in a consistent way. In particular we would draw attention

Table 3: Fuzzy Pairwise comparison matrices used in examples (source [1])

$$\tilde{\mathbf{C}}_1 = \begin{bmatrix} \tilde{1} & \tilde{3} & \tilde{9} \\ \tilde{3}^{-1} & \tilde{1} & \tilde{5} \\ \tilde{9}^{-1} & \tilde{5}^{-1} & \tilde{1} \end{bmatrix} \qquad \tilde{\mathbf{C}}_2 = \begin{bmatrix} \tilde{1} & \tilde{3} & \tilde{5}^{-1} \\ \tilde{3}^{-1} & \tilde{1} & \tilde{9} \\ \tilde{5} & \tilde{9}^{-1} & \tilde{1} \end{bmatrix}$$

$$\tilde{\mathbf{C}}_4 = \begin{bmatrix} \tilde{1} & \tilde{7}^{-1} & \tilde{3} \\ \tilde{7} & \tilde{1} & \tilde{9} \\ \tilde{3}^{-1} & \tilde{9}^{-1} & \tilde{1} \end{bmatrix} \qquad \tilde{\mathbf{C}}_3 = \begin{bmatrix} \tilde{1} & \tilde{9}^{-1} & \tilde{7} \\ \tilde{9} & \tilde{1} & \tilde{3}^{-1} \\ \tilde{7}^{-1} & \tilde{3} & \tilde{1} \end{bmatrix}$$

$$\widetilde{\mathbf{W}} = \begin{bmatrix} \tilde{1} & \tilde{3} & \tilde{7} & \tilde{5} \\ \tilde{3}^{-1} & \tilde{1} & \tilde{9} & \tilde{3} \\ \tilde{7}^{-1} & \tilde{9}^{-1} & \tilde{1} & \tilde{3}^{-1} \\ \tilde{5}^{-1} & \tilde{3}^{-1} & \tilde{3} & \tilde{1} \end{bmatrix}$$

to the extremely strong correlation between the kernel values and imprecision. It is apparent that neither requirement 2 nor requirement 3 of section 3 are met.

5 Discussion

There are reports of the Fuzzy AHP being successfully applied both in [1] and elsewhere, further research is needed to investigate why this is so in the light of these results. These results also appear to be an example of the practical problems encountered when using standard fuzzy arithmetic in applications anticipated by Klir [3].

References

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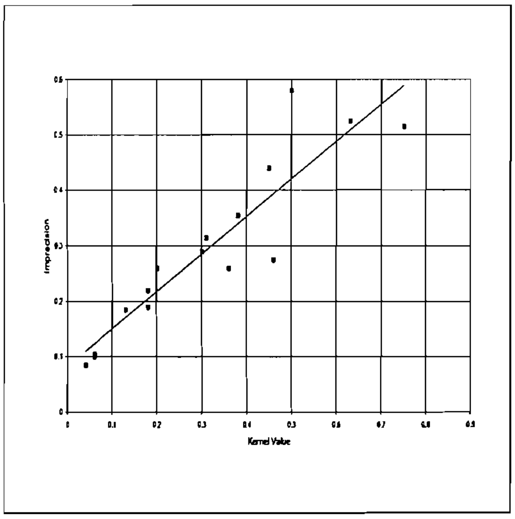

Figure 1: Imprecision and Kernel Value based on the row sum method using data from [1]

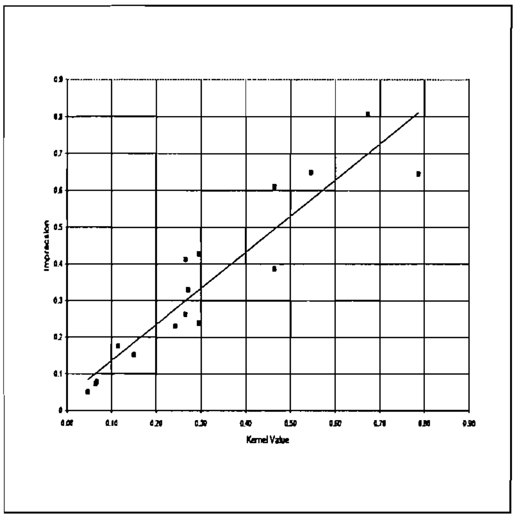

Figure 2: Kernel Value plotted against width of support using the normalised geometric mean method based on data from [1]