A general framework for probabilistic approaches to fuzzy quantification

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Abstract

This framework allows the description of previous probabilistic models and also the definition of new ones, all of them endowed with a semantic interpretation based on a "voter's profile". The methods within the framework fulfil a number of important and adequate properties of interest for fuzzy quantification and can also deal with very different types of quantifiers, as comparative and exception ones.

Keywords: Fuzzy quantifiers, Generalized Quantifiers, Mass Assignments.

1 Introduction

Evaluation of quantified sentences is a topic that has been widely dealt with in literature [2, 5, 7, 9, 10], since use of quantified sentences in fields like fuzzy control, fuzzy expert systems development, decision-making, complex fuzzy queries in databases or information retrieval offers a significant increase in the functionality and aplicability of systems.

Following [10], most of these approaches aim to model simple expressions involving just one fuzzy property (type 1 sentences, as "more than 5 TV sets are *large*") or two (type 2 sentences as "more than a half *cheap* TV sets are *small"*) and absolute or relative quantifiers that, respectively, represent a fuzzy quantity (*"five*") or proportion ("*a half*"). In spite of this simplification of the problem, most of these approaches fail to fulfil a number of important properties for type II sentences, thus showing non-plausible behaviour ([7]). Furthermore, other type of propositions (involving more than two properties) and/or other type of quantifiers (e.g., comparative or exception ones [8], as in *"There are more tall members than blondes"* and *"All but 3 tall members are blondes"*, respectively) cannot be straightforwardly addressed by these methods. Only a few proposals exhibit more adequate behaviours and are capable of dealing with more complex quantifiers: the trivalued by [6] and the probabilistic ones by [2, 4]

In this paper a general model that frames these probabilistic approaches for quantified sentences and allows definition of new ones is presented. The study of these methods and the developement of new proposals can be carried out under this framework. Following the scheme in [6], evaluation of quantified sentences is presented under the approach of *fuzzy predeterminers*, that allows modeling sentences including comparative and exception quantifiers. The underlying semantic interpretation of our model is the one based in the voting model [1], which is briefly described in the next section.

2 Description of the probabilistic frame

Let $E = \{e_1, e_2, \ldots, e_R\}$ be a referential set of individuals. Let $X = \{x_1, x_2, \ldots, x_R\}$ be the set of degrees to which every $e_r \in E$, $r = 1, \ldots, R$, fulfils fuzzy property $\widetilde{X}$, described by fuzzy set $\mu_{\widetilde{X}}$. Under the voting model interpretation ([1]), fuzzy set $\mu_{\widetilde{X}}$ is described within context E as a summarization of the decision of a set of voters that were asked for crisp opinions on the $e_r \in E$ that fulfil $\widetilde{X}$. Furthermore, under intuitive coherence assumptions, each voter can be assigned an α-level cut that defines its voting as the α-cut $\left(\widetilde{X}\right)_{\alpha} = \{e_r \in E / x_r \geq \alpha\}$. Probability distribution $P(\alpha)$ of choosing a given level α is assumed to be uniformly distributed among the voters.

2.1 Mechanisms for the evaluation of quantified sentences

Let us firstly present some preliminar definitions and examples, following [6, 8]).

Definition 1 (Determiner) *[6] An n-ary determiner on a referential set E is defined as a function* $D : \wp(E)^n \longrightarrow \{0,1\}$, *$\wp(E)$ being the power set of E.*

Under this definition, determiners correspond to crisp quantifiers. The degree of fulfilment for quantified type II ($n = 2$) expressions like "D X_1 are X_2", expressed as $D(X_1,X_2)$, can be defined as the degree of truth of logical expressions, as shown in the following examples:

$$\begin{aligned}
All\,(X_1,X_2) &:= X_1 \subseteq X_2 \\
Allbut3\,(X_1,X_2) &:= |X_1| - |X_1 \cap X_2| = 3 \\
Between4and6\,(X_1,X_2) &:= |X_1 \cap X_2| \in [4,6] \\
GreaterorEq50\%\,(X_1,X_2) &:= \begin{cases} \frac{|X_1 \cap X_2|}{|X_1|} \geq 0.50 & X_1 \neq \emptyset \\ v \in \{0,1\} & X_1 = \emptyset \end{cases}
\end{aligned}$$

Definition 2 (Fuzzy determiner) *[6] An n-ary fuzzy determiner on a referential $E \neq \emptyset$ is defined as a function* $\widetilde{D} : \widetilde{\wp}(E)^n \longrightarrow [0,1]$, *with $\widetilde{\wp}(E)$ being the fuzzy power set of E.*

Fuzzy determiners correspond to fuzzy quantifiers.

Definition 3 (Fuzzy Pre-determiner) *[6] An n-ary fuzzy predeterminer on a base set $E \neq \emptyset$ is defined as a function* $D : \wp(E)^n \longrightarrow [0,1]$.

Some examples of predeterminers and its logical-based definitions are:

$$\begin{aligned}
AboutBetween4and6\,(X_1,X_2) &:= T_{2,4,6,8}\,(|X_1 \cap X_2|) \\
About50\%orMore\,(X_1,X_2) &:= \begin{cases} S_{0.3,0.5}\left(\frac{|X_1 \cap X_2|}{|X_1|}\right), X_1 \neq \emptyset \\ v \in [0,1], X_1 = \emptyset \end{cases}
\end{aligned} \tag{1}$$

where $T_{2,4,6,8}$ is a trapezoid with support $[2,8]$ and kernel $[4,6]$, and $S_{0.5,0.8}$ is the usual Zadeh's S function.

The usefulness of defining fuzzy pre-determiners is for making simpler the evaluation of complex quantified sentences. Use of predeterminers can be understood as an intermediate step between crisp and fuzzy quantifiers. By means of defining a transformation mechanism for converting predeterminers into fuzzy determiners, quantification can be addressed even for complex sentences (e.g., involving exception quantifiers like "*All but three*"). In [6], a number of fuzzification mechanisms are presented for trivalued crisp representatives. The new one we briefly present in what follows is an adequate general solution for bivalued α-cuts.

2.2 A general probabilistic framework

Let D be an n-ary fuzzy pre-determiner. An n-ary probabilistic fuzzy determiner $\widetilde{D}$ associated to D is defined as:

$$\widetilde{D}(X_1,\ldots,X_n) := \int_0^1 \cdots \int_0^1 (D(X_1)_{\alpha_1},\ldots,(X_n)_{\alpha_n})\,P(\alpha_1,\ldots,\alpha_n)\,d\alpha_1 \ldots d\alpha_n \tag{2}$$

where $P(\alpha_1,\ldots,\alpha_n)$ is a probability density that describes the relationship existing among the different α-cut levels for the different properties $X_1,\ldots,X_n$. Under the voting model interpretation, $P(\alpha_1,\ldots,\alpha_n)$ defines the relationship among the different levels that are chosen by the voters for the different properties. In this way, value $\widetilde{D}(X_1,\ldots,X_n)$ represents the mean of the values provided by the voters. Different underlying semantic interpretations that produce different numeric results arise from the use of different probability densities. Probability density $P(\alpha_1,\ldots,\alpha_n)$ can be interpreted as the pattern or profile that describes voters' or experts' behaviour when asked to build fuzzy sets. This density is strongly related to the features of the field where the quantification problem is stated and should be defined in advance. Some a priori definitions (or *profiles*) of P with its associated meaning can be done under this framework. Thus different proposals for probabilistic evaluation of fuzzy quantified sentences can be directly described and semantically interpreted (e.g., the ones by [2] and [4]):[1] By taking a Dirac delta function as the probability density $P(\alpha_1,\alpha_2) = \delta(\alpha_1 - \alpha_2)\,\forall \alpha_1,\alpha_2 \in [0,1]$ in (2) (i.e., all voters take the same α-cut level for X_1 and X_2), we have a *maximal dependence profile* between X_1 and X_2. In this case, (2) becomes

$$\widetilde{D}_1\,(X_1,\ldots,X_n) := \int_0^1 D((X_1)_\alpha,\ldots,(X_n)_\alpha)\,d\alpha \tag{3}$$

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[1] For the sake of clarity, only the case $n = 2$ is presented.

that corresponds to the probabilistic model in [2]. Since the set of α-cuts on X_1,X_2 is usually finite, expression (3) becomes $\widetilde{D}_1(X_1,X_2)=\sum_{\alpha_i} D\left((X_1)_{\alpha_i},(X_2)_{\alpha_i}\right) m(\alpha_i)$, where $\alpha_0 = 1 > \alpha_1 > \ldots > \alpha_k$ denote the different membership values of the elements in E to fuzzy sets X_1,X_2, and $m(\alpha_i) = \alpha_i - \alpha_{i+1}$, $i = 0,\ldots,k$ with $\alpha_{k+1} = 0$.

By taking $P(\alpha_1,\alpha_2) = 1, \forall \alpha_1,\alpha_2 \in [0,1]$, an *independence profile* between X_1 and X_2 is obtained, since this assumes that the α-cuts that are chosen for X_1 are independent of those for X_2. For this case, (2) becomes

$$\widetilde{D}_2(X_1,X_2) := \int_0^1 \int_0^1 D\left((X_1)_{\alpha_1},(X_2)_{\alpha_2}\right) d\alpha_1 d\alpha_2 \tag{4}$$

which is the model described in [4]. For the finite case, (4) becomes $\widetilde{D}_2(X_1,X_2)=\sum_{\alpha_i}\sum_{\alpha_j} D\left((X_1)_{\alpha_1},(X_2)_{\alpha_2}\right) m(\alpha_i) m(\alpha_j)$, where α_i and α_j denote the membership values of the elements in E to X_1 and X_2, respectively.

Example 1: Let us consider the following sentence: *About 50% or more tall men are blondes*, where predeterminer D ="about 50% or more", and membership of the elements on universe E ="men" to fuzzy sets "tall" and "blondes" take, respectively, the following values: $X_1 = \{1,1,0.8,0.3,1\}$, $X_2 = \{0.8,0.9,0.7,0.2,0.1\}$, and the predeterminer is defined in (1).

The results that are obtained for the two previously presented profiles are, respectively, $\widetilde{D}_1(X_1,X_2) = 0.81$ and $\widetilde{D}_2(X_1,X_2) = 0.79$. The fact that both of the profiles produce very similar results is consistent with the situation the example is considering. Under a "commonsense" interpretation of X_1 and X_2, it can be said that only the elements $\{e_1,e_2,e_3,e_5\}$ have a significant membership to X_1 and therefore should be taken much more into account than e_4. From them, $\{e_1,e_2,e_3\}$ have a simultaneous significant membership to X_2 (that is, three elements out of four, which is more than 50%) and therefore a high value for $\widetilde{D}(X_1,X_2)$ should be expected.

Example 2: Let us consider now the sentence: *Almost all tall men are blondes*, where predeterminer D ="Almost all", and X_1, X_2 take the following values: $X_1 = \{1,0.8,0.6,0.4\}$, $X_2 = \{0.8,0.6,0.4,0.2\}$,

$$D(X_1,X_2)=\begin{cases} \max\left\{4\left(\frac{|X_1\cap X_2|}{|X_1|}\right)-3,0\right\} & X_1\neq\emptyset \\ 1 & X_1=\emptyset \end{cases}$$

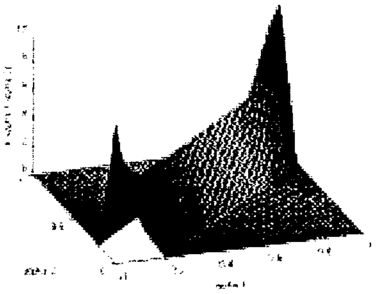

Figure 1: $P(\alpha_1,\alpha_2)$ for the "approximate dependence profile", with $\delta = 0.2$

The results that are obtained for the two previously presented profiles are, respectively, $\widetilde{D}_1(X_1,X_2) = 0.2$ and $\widetilde{D}_2(X_1,X_2) = 0.44$. This example is subject to modelling/interpretation since elements $\{e_1,e_2\}$ have a significant membership to X_1, whilst $\{e_3,e_4\}$ lie on the half range between full membership and nonmembership. Results for this case are strongly dependent on the underlying interpretation that is expressed by the profiles, as these numeric results show.

2.3 A new approach for evaluating quantified sentences

An approximated dependence when choosing the α-cut levels can be modelled, in order to get a sort of flexibility for this election, ranging between maximal dependence and total independence. This way, once an α-level α_2 was selected for X_2, voter is allowed to select an α-level α_1 for X_1, that can be linguistically defined as "*approximately* α_2". This profile relaxes the maximal dependence one.

A triangular piecewise probability density $P(\alpha_1,\alpha_2)$ consistent with this interpretation can be defined as:

$$P(\alpha_1,\alpha_2)=\begin{cases} h_1-\min\left(h_1,h_1\frac{|\alpha_1-\alpha_2|}{\delta}\right) & 0\leq\alpha_2\leq\delta \\ \frac{1}{\delta}-\min\left(\frac{1}{\delta},\frac{|\alpha_1-\alpha_2|}{\delta^2}\right) & \delta\leq\alpha_2\leq 1-\delta \\ h_2-\min\left(h_2,h_2\frac{|\alpha_1-\alpha_2|}{\delta}\right) & 1-\delta\leq\alpha_2\leq 1 \end{cases}$$

where $h_1 = -\frac{2}{\alpha_2^2-2\alpha_2\delta-\delta^2}\delta$, $h_2 = \frac{2}{1-2\alpha_2+\alpha_2^2+\delta^2}\delta$, and $0 \leq \delta \leq 0.5$ define a flexibility parameter, which is half the base of the triangular function. Figure 1 shows this function for value $\delta = 0.2$.

For example, estimating (2) by using the rectangle numerical method we have:

$$\widetilde{D}_3(X_1,X_2) = \\ = h^2 \sum_{i,j=0}^{l-1} D((X_1)_{ih+\frac{h}{2}},(X_2)_{jh+\frac{h}{2}})P(ih+\frac{h}{2},jh+\frac{h}{2})$$

where l is the number of discretization intervals, h is the discretization step, and P is approached in the middle point of each interval.

Example: For the examples in the previous section, we have for $\delta = 0.2$, $\widetilde{D}_3\,(X_1,X_2) \approx 0.81$ in the first case and $\widetilde{D}_3\,(X_1,X_2) \approx 0.30$ in the second. As expected, the first result keeps being consistent with the other two profiles, while the second value lies in between them.

3 Analysis of properties for the framework

In ([2], [3], [7]) some properties of interest for quantification models are defined. Some of the most important properties[2] that are fulfiled by this framework are: *correct generalization of crisp expressions*, *extern negation*, *monotonicity*, *local monotonicity specificity in quantifiers*, *independence with respect to the permutation of elements in quantitative predeterminers*, etc. The model under "maximal dependence" profile fulfils *internal meets and conservativity*, whilst "independence model" profile fulfils *antonyms* and *dual* properties. Both of the profiles fulfil the *induced operators* property. In general, *continuity* property is not fulfiled (although it is very likely that continuity be fulfiled for the most usual predeterminers under the independence profile), neither *coherence with logic* (as a result of the probabilistic interpretation of the model).

Complexity for the exact resolution algorithms is $O(n \log n)$ for maximal dependence profile (approximate results can be obtained in time $O(n)$) and $O(n^2)$ for independence profile). Furthermore, by using the voting model interpretation and defining different profiles a simple and understandable semantics is associated to the probabilistic framework, and this is an interesting feature that should be also pointed out.

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