Sampling inspections by attributes which are based on soft quality standards

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Abstract
This paper tries to establish a framework for sampling inspections by attributes based on soft quality standards in order to pay more attention to some shortcomings in classical modelling concerning the acceptability of inspection lots. As a suggestion such sampling inspections are described in terms of a statistical test with fuzzy hypothesis and fuzzy data, extending the classical framework of sampling inspections by attributes. Furthermore the operating characteristics will be explored.
KEYWORDS: Inspection by attributes, variable of measurement, single sampling plan, sampling inspection by attributes.

0 Introduction

A well developed framework is available to controllers when dealing with sampling inspections by attributes ([5], [8]). Within statistical terminology sampling inspections by attributes can be described as one-sided statistical tests about the portion of the defective units in the inspection lot under consideration. An inspection lot is defined to be acceptable if the portion of the defective units does not exceed a given tolerance value, that means that the nullhypothesis is valid. Recently Arnold has directed the attention to some strictness of this approach ([1], [2]). Firstly the costumer might be satisfied with a test which satisfies the constraints by the errors of first and second kind not exactly if he gets some compensation e.g. a smaller sample size ([1]). Secondly, defining the acceptability of an inspection lot, it should be taken into account to what amount the portion of defective units deviates from a given tolerance value. Concerning both points Arnold has offered some extension of classical test theory by using fuzzy set theory.

This paper is also a contribution to the integration of fuzzy set theory into sampling inspections by attributes. We shall focus on the shortcomings of the classical approach caused by the usual formulation of the quality standard underlying each inspection by attributes. When looking for a suitable standard we are often faced with the so called problem of adequacy. We have to find indicators and boundaries for them in order to measure the quality of the units adequately. In many cases the fixing of the boundaries seems to be arbitrary. Moreover, very heterogeneous kinds of qualities are reduced to a common denominator since it is of no importance whether the deviation of a unit from the quality standard is small or large. This can lead to unsatisfactory assessments of the quality of a lot.

In order to get by these problems the literature on statistical quality control offers the method of sampling inspections by variables which leads to parameter tests on the fractions defective in the inspection lots (cf. [8]). Unfortunately, this framework does not reduce the shortcomings of sampling inspections by attributes caused by the problem of adequacy. The quality of a lot is still characterized by the fraction defective which relies on a classical formulation of a quality standard.

The aim of this paper is to generalize the classical framework of sampling inspections by attributes to sampling inspections based on soft quality standards. That means to go over to quality standards which may differentiate between grades of quality with the antipodean extreme grades of strict quality and full defectiveness. Then the amount of deviation from strict quality could be

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taken into consideration to define the quality of an inspection lot. The approach differs from the Hryniewicz's proposal to overcome the shortcomings caused by the strict quality standards (cf. [4]). His crucial idea is to relax the qualification of units of the lot w.r.t. a given strict standard, allowing imprecise evaluations to express how the units fulfill the quality requirements.

The paper is organized as follows. After introducing some basic notations, we want to install in section 2 the general setting for an inspection of lots by attributes based on soft standards. In section 3 we adapt the classical single sampling plans to the inspections built on soft standards. Then we may describe this general sampling inspection by attributes with single sampling plans as a statistical test, and the operating characteristics will be explored completely.

1 Notations and preliminaries

Let Ω be a non-void set. The $\alpha-$cuts of a fuzzy subset $\widetilde{R}$ of Ω will be denoted by $[\widetilde{R}]^\alpha$ ($\alpha \in]0,1]$). Throughout the paper we shall focus our attention to the space $F(\mathbb{R})$ consisting of all fuzzy subsets $\widetilde{A}$ of $\mathbb{R}$ with non-void, convex, compact $\alpha-$cuts $[\widetilde{A}]^\alpha$ ($\alpha \in]0,1]$) satisfying $\min[\widetilde{A}]^\alpha = \max[\widetilde{A}]^1$ for each $\alpha \in]0,1]$. This fuzzy subset is uniquely described by the function $r_{\widetilde{A}} : [0,1] \to \mathbb{R}$ with $r_{\widetilde{A}}(0) := 1$, $r_{\widetilde{A}}(1) := 0$ and $r_{\widetilde{A}}(\alpha) := \max[\widetilde{A}]^\alpha$ otherwise.

Using Zadeh's extension principle we may endow $F(\mathbb{R})$ with the well-defined operations $\oplus_F$ of addition and $\lambda \odot_F$ scalar mulitiplications with non-negative real numbers λ. Obviously the neutral element w.r.t. $\oplus_F$ is the fuzzy subset with the indicator mapping of $\{0\}$ as membership function. This fuzzy subset will be denoted by $\widetilde{0}$.

Several suggestions to extend the ordinary $\leq -$relation on $\mathbb{R}$ to $F(\mathbb{R})$ can be found in literature (e.g. [3], [7]). We shall use the relation $\preceq$, defined to consist of all $(\widetilde{A},\widetilde{B}) \in F(\mathbb{R})^2$ such that $r_{\widetilde{A}}(\alpha) \leq r_{\widetilde{B}}(\alpha)$ holds for each $\alpha \in]0,1[$, and write $\widetilde{A} \preceq \widetilde{B}$ if $(\widetilde{A},\widetilde{B}) \in \preceq$. The choice of this relation is justified by the pretty interpretation which can be revealed in the context of sampling inspections by attributs.

Throughout this paper the number of a finite set A will be denoted by $\sharp A$.

2 General inspections by attributes

From now on let Ω denote a lot containing N units of product. It may be supplied by a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where P stands for the relative frequency on the powerset $\mathcal{F}$ of Ω.

Usually inspection by attributes means to accept or reject the lot w.r.t. a list of quality characteristics and requirements which define the quality standard. If e.g. the list consists of m quality characteristics and requirements, we can represent the standard of defective units of product by a subset Q of a Cartesian product $\bigtimes_{i=1}^{m} \Omega_i$ of non-void sets $\Omega_1, ..., \Omega_m$. This quality standard induces the subset R of the defective units in the lot Ω. Then the **quality of the lot Ω w.r.t. Q** is defined as the fraction defective $\frac{\sharp R}{N} =: p$, which is unknown.

As it had been pointed out in the introduction it is often not adequate to define a strict standard. In the following we want to suggest an extension of the classical inspections by attributes. The crucial point is to relax the restrictions of a classical quality standard, defining a soft quality standard. Instead of the subset Q of $\bigtimes_{i=1}^{m} \Omega_i$ we use a fuzzy subset $\widetilde{Q}$ of $\bigtimes_{i=1}^{m} \Omega_i$ to represent the quality standard. This new standard induces a fuzzy subset $\widetilde{R}$ of Ω. But now we have to define the quality of Ω w.r.t. $\widetilde{Q}$, where no canonical way is offered.

The values of the membership function $\mu_{\widetilde{Q}}$ can be understood as grades of defectiveness. Thus we can classify hierarchically the units of lots w.r.t. these grades of defectiveness. Each $\alpha-$cut $[\widetilde{R}]^\alpha$ gathers all the units of Ω with a grade of defectiveness which is at least α. Observe that in the case of the strict standard Q we obtain R as the unique class of defectiveness. Since the portion $\frac{\sharp R}{N}$ defines the quality of Ω w.r.t. Q, we intuitively describe the quality of Ω w.r.t. $\widetilde{Q}$ by the simultaneous fractions defective $\{\frac{\sharp[\widetilde{R}]^\alpha}{N} \mid \alpha \in]0,1]\}$. A very simple way to compress these quantities into a formal expression is the fuzzy subset $\widetilde{\vartheta}$ from $F(\mathbb{R})$, characterized by $r_{\widetilde{\vartheta}}(\alpha) := \frac{\sharp[\widetilde{R}]^\alpha}{N}$ for $\alpha \in]0,1[$. It will be called the **quality of the lot Ω w.r.t. $\widetilde{Q}$**. The unknown quality $\widetilde{\vartheta}$ possibly

takes values in the parameter space Θ_N consisting of all $\widetilde{\vartheta} \in F(\mathbb{R})$ such that the range of $r_{\widetilde{\vartheta}}$ is contained in $\{\frac{i}{N} \mid i \in \{0, ..., N\}\}$.
In the next step we want to fix when the quality of the lot may be regarded as acceptable. The idea is, analogously to the inspection of Ω w.r.t. Q, to set the portions $\frac{\sharp[\widetilde{R}]^\alpha}{N}$ ($\alpha \in]0,1]$) respectively upper values t_α ($\alpha \in]0,1]$) of tolerance. Formally, this means to choose some $\widetilde{\vartheta}_l \in \Theta_N$ such that the quantity $\widetilde{\vartheta}$ of the lot is acceptable if $\widetilde{\vartheta} \preceq \widetilde{\vartheta}_l$. Therefore $\widetilde{\vartheta}_l$ is called **value of tolerance**. The acceptable qualities from Θ_N are gathered in the set Θ_{N0}.
Summarizing the discussion, the quintuple $(\widetilde{Q}, \widetilde{R}, \Theta_N, \widetilde{\vartheta}_l, \Theta_{N0})$ defines the **inspection by attributes based on the soft quality standard $\widetilde{Q}$**.
As in the case of classical inspections by attributes sampling inspection will be treated like a statistical parameter test with pair of hypothesis $H_0 : \widetilde{\vartheta} \in \Theta_{N0}, \quad H_1 : \widetilde{\vartheta} \notin \Theta_{N0}$. The critical and acceptance regions will be based on random samples from the lot by investigating the qualities of the elements in the samples. In order to construct these regions we need a suitable variable of measurement. As a plausible requirement this variable should take the quality $\widetilde{\vartheta}$ of the lot Ω as its average value. This may be achieved by $\widetilde{Y}^{\widetilde{R}}(\omega) \in F(\mathbb{R})$, characterized by $r_{\widetilde{Y}^{\widetilde{R}}(\omega)}(\alpha) := 1$ if $\omega \in [\widetilde{R}]^\alpha$, and $r_{\widetilde{Y}^{\widetilde{R}}(\omega)}(\alpha) := 0$ otherwise ($\alpha \in]0,1[$). This mapping $\widetilde{Y}^R : \Omega \to F(\mathbb{R})$ will be named the **variable of measurement** w.r.t. $(\widetilde{Q}, \widetilde{R}, \Theta_N, \widetilde{\vartheta}_l, \Theta_{N0})$. It may be viewed as a random variable over $(\Omega, \mathcal{F}, \mathrm{P})$, with the image measure of P on the powerset of $F(\mathbb{R})$ as its distribution.

3 Single sampling plans

In most cases we have to content ourselves with applying an inspection by attributes to a sample drawn from the inspection lot. Therefore we have to find sampling plans which fix the decision rule to accept or reject the inspection lot on basis of such samples. This will be the subject of this section, restricting ourselves to sampling plans which extends the single sampling plans of classical inspections by attributes.

Let a sample $(\omega_1, ..., \omega_n)$ be drawn from Ω. Then by construction $r_{\widetilde{Y}^{\widetilde{R}}(\omega_1) \oplus_F ... \oplus_F \widetilde{Y}^{\widetilde{R}}(\omega_n)}(\alpha)$ counts the elements of the sample which belong to the class of defectiveness $[\widetilde{R}]^\alpha$. Now the analogy to classical single sampling plans is obvious:
In order to define a lot to be acceptable in the view of $\widetilde{Y}^{\widetilde{R}}(\omega_1), ..., \widetilde{Y}^{\widetilde{R}}(\omega_n)$ we demand that $r_{\widetilde{Y}^{\widetilde{R}}(\omega_1) \oplus_F ... \oplus_F \widetilde{Y}^{\widetilde{R}}(\omega_n)}(\alpha) \leq C_\alpha$ ($\alpha \in]0,1[$) hold simultaneously, where C_α ($\alpha \in]0,1[$) are given numbers. This means to select a $\widetilde{C} \in F(\mathbb{R})$ with $\widetilde{0} \preceq \widetilde{C}$ such that $\widetilde{Y}^{\widetilde{R}}(\omega_1) \oplus_F ... \oplus_F \widetilde{Y}^{\widetilde{R}}(\omega_n) \preceq \widetilde{C}$ should be satisfied to accept the lot.
So we define a **single sampling plan** w.r.t. the inspection $(\widetilde{Q}, \widetilde{R}, \Theta_N, \widetilde{\vartheta}_l, \Theta_{N0})$ as a pair $(n, \widetilde{C})$ from $\{1, ..., N\} \times F(\mathbb{R})$ with $\widetilde{0} \preceq \widetilde{C}$. Each single sampling plan $(n, \widetilde{C})$ induces a statistical test with hypotheses $H_0 : \widetilde{\vartheta} \in \Theta_{N0}$, $H_1 : \widetilde{\vartheta} \notin \Theta_{N0}$ and acceptance region $A_{n,\widetilde{C}}$ consisting of all $(\widetilde{x}_1, ..., \widetilde{x}_n) \in F(\mathbb{R})^n$ with $\widetilde{x}_1 \oplus_F ... \oplus_F \widetilde{x}_n \preceq \widetilde{C}$. Such a test defines a **sampling inspection by attributes based on the soft standard $\widetilde{Q}$**.
The power of sampling inspections is expressed by their operating characteristics. Our aim is to investigate them, restricting ourselves to samples without replacement. We shall obtain an extension of the classical result in terms of multivariate hypergeometric distributions, which is, however, more complicated since the structure of the quality of an inspection lot is more complex than within the classical setting.
Let us associate each $\widetilde{\vartheta} \in \Theta_N$ with the set of $D(\widetilde{\vartheta})$ gathering the $m_{\widetilde{\vartheta}}$ different discontinuity points $\alpha_{1\widetilde{\vartheta}} < ... < \alpha_{m_{\widetilde{\vartheta}}\widetilde{\vartheta}}$ of $r_{\widetilde{\vartheta}}$. Furthermore let $\widetilde{R}_{\widetilde{\vartheta}}$ denote any fuzzy subset of the lot Ω with $\max[\widetilde{\vartheta}]^\alpha = \dfrac{\sharp[\widetilde{R}_{\widetilde{\vartheta}}]^\alpha}{N}$ for $\alpha \in]0,1]$. It may be viewed as the fuzzy subset of defective units in the lot if $\widetilde{\vartheta}$ is the true quality. Observe that this interpretation does not depend on the choice of $\widetilde{R}_{\widetilde{\vartheta}}$. The discontinuity points of $r_{\widetilde{\vartheta}}$ are just the outcomes of the membership function $\mu_{\widetilde{R}_{\widetilde{\vartheta}}}$ of $\widetilde{R}_{\widetilde{\vartheta}}$. Using these notations, $H(N, n, \widetilde{\vartheta})$ denotes the Dirac measure at the real number 0 if $\widetilde{\vartheta} = \widetilde{0}$. Otherwise this symbol will be used for the $m_{\widetilde{\vartheta}}-$variate hypergeometric distribution with parameters $N, n, Np_{\widetilde{\vartheta}}(\alpha_{1\widetilde{\vartheta}}), ..., Np_{\widetilde{\vartheta}}(\alpha_{m_{\widetilde{\vartheta}}\widetilde{\vartheta}})$, where $p_{\widetilde{\vartheta}}(\alpha_{i\widetilde{\vartheta}}) := \sharp\mu^{-1}_{\widetilde{R}_{\widetilde{\vartheta}}}(\{\alpha_{i\widetilde{\vartheta}}\})$. Note that the dimension of the distribution is linked with the

number of discontinuity points of $r_{\widetilde{\vartheta}}$.

Theorem 3.1 *Let $(n, \widetilde{C})$ be a single sampling plan w.r.t. $(\widetilde{Q}, \widetilde{R}, \Theta_N, \widetilde{\vartheta}_l, \Theta_{N0})$, and additionally let $L_{N,n,\widetilde{C}} : \Theta_N \to \mathbb{R}$ denote the operating characteristic of the induced statistical test. Then $L_{N,n,\widetilde{C}}(\widetilde{0}) = 1$, and for $\widetilde{\vartheta} \neq \widetilde{0}$ we obtain*

$$L_{N,n,\widetilde{C}}(\widetilde{\vartheta}) = H(N, n, \widetilde{\vartheta}) \ (\bigcap_{\alpha \in D_{\widetilde{\vartheta}} \setminus \{0\}} A_\alpha),$$

where A_α consists of all $(z_1, ..., z_{m_{\widetilde{\vartheta}}}) \in \mathbb{R}^{m_{\widetilde{\vartheta}}}$ such that the inequality $\sum_{\substack{k=1 \\ \alpha \leq \alpha_{k\widetilde{\vartheta}}}}^{m_{\widetilde{\vartheta}}} z_k \leq \max[\widetilde{C}]^\alpha$ holds $(\alpha \in D_{\widetilde{\vartheta}} \setminus \{0\})$.

One consequence of Theorem 3.1 is that the operating characteristics are only dependent on the chosen single sampling plan. Therefore we may call the mapping $L_{N,n,\widetilde{C}}$ the **operating characteristic of the quality of an inspection lot of size N based on the single sampling plan** $(n, \widetilde{C})$.
It is known that within the framework of classical sampling inspections by attributes with single sampling plans the operating characteristics are nonincreasing. This result may be preserved.

Proposition 3.2 *Let $L_{N,n,\widetilde{C}}$ be the operating characteristic of the quality of an inspection lot of size N based on the single sampling plan $(n, \widetilde{C})$. Then $L_{N,n,\widetilde{C}}(\widetilde{\vartheta}_1) \geq L_{N,n,\widetilde{C}}(\widetilde{\vartheta}_2)$ if $\widetilde{\vartheta}_1 \preceq \widetilde{\vartheta}_2$.*

When selecting a single sampling plan we can restrict ourselves to consider only finite many single sampling plans.

Proposition 3.3 *Let $(n, \widetilde{C})$ be some single sampling plan w.r.t. $(\widetilde{Q}, \widetilde{R}, \Theta_N, \widetilde{\vartheta}_l, \Theta_{N0})$.
Then there exists some $\widetilde{C}^* \in F^{no}_{cocrf}(\mathbb{R})$ such that the single sampling plan $(n, \widetilde{C}^*)$ satisfies the following properties*

.1 $r_{\widetilde{C}^*}(\alpha) \in \{0, ..., n\}$ *for* $\alpha \in]0, 1]$, *and additionally* $\mu_{\widetilde{C}^*}(\mathbb{R}) \setminus \{0\} \subseteq D(\widetilde{\vartheta}_l) \ \cup \{1\}$.

.2 $L_{N,n,C^*}(\widetilde{\vartheta}) \leq L_{N,n,C}(\widetilde{\vartheta})$ *for all* $\widetilde{\vartheta} \in \Theta_N$.

.3 $\min_{\widetilde{\vartheta} \preceq \widetilde{\vartheta}_l} L_{N,n,C^*}(\widetilde{\vartheta}) = \min_{\widetilde{\vartheta} \preceq \widetilde{\vartheta}_l} L_{N,n,C}(\widetilde{\vartheta})$

4 Final remarks

We have introduced a framework of sampling inspections by attributes which generalizes the classical one by incorporating soft quality standards for the units of the inspection lots. So the obtained greater flexibility for planning might be utilized to represent the quality and acceptability of inspection lots more adequately than the classical approach. These advantages are, however, contrasted by the practical difficulty to evaluate operating characterists which are more complicated than in the classical context.
The author has also provided a method to select a suitable single sampling. It extends the classical one to find single sampling plans of given strength (cf. [6]).

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