

On the entropy on the Lukasiewicz square

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Abstract

The unit square is regarded as a special kind of dynamical system. It is proved that the square is an MV-algebra, hence the recent results on MV-algebra entropy theory can be applied to it.

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1 Introduction

We shall consider the square $M = [0, 1]^2$ as a partially ordered set with the ordering

$$A_1 = (x_1, y_1) \leq (x_2, y_2) = A_2 \iff x_1 \leq x_2, y_1 \geq y_2$$

With respect to this ordering, M is a lattice with the operations $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2)$, $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2)$ and the least element $(0, 1)$ and the greatest element $(1, 0)$. In [1] the following two binary operations \oplus, \odot have been introduced:

$$A_1 \oplus A_2 = (x_1, y_1) \oplus (x_2, y_2) = (x_1 \oplus x_2, y_1 \odot y_2),$$

$$A_1 \odot A_2 = (x_1, y_1) \odot (x_2, y_2) = (x_1 \odot x_2, y_1 \oplus y_2),$$

where

$$x_1 \oplus x_2 = (x_1 + x_2) \wedge 1, x_1 \odot x_2 = (x_1 + x_2 - 1) \vee 0.$$

In [5] the probability on M has been defined as a function $P : M \rightarrow [0, 1]$ satisfying the following conditions:

$$P((1, 0)) = 1, P((0, 1)) = 0,$$

$$A_1 \odot A_2 = (0, 1) \implies P(A_1 \oplus A_2) = P(A_1) + P(A_2),$$

$$A_n \nearrow A \iff P(A_n) \nearrow P(A).$$

It has been shown in [5] that to any probability $P : M \rightarrow [0, 1]$ there exists $\alpha \in [0, 1]$ such that

$$P((x, y)) = (1 - \alpha)x - \alpha y + \alpha.$$

By a dynamical system we understand a couple (P, T) such that $P : M \rightarrow [0, 1]$ is a probability and $T : M \rightarrow M$ is a measure-preserving map, i.e.

$$(i) \quad a = b \oplus c \implies T(a) = T(b) \oplus T(c),$$

$$(ii) \quad T((1, 0)) = (1, 0),$$

$$(iii) \quad P(T(x)) = P(x)$$

for every $x \in M$. The aim of this paper is to construct the entropy of the dynamical system as an analogy of the Kolmogorov - Sinaj entropy [9]. We propose it in Section 2 as the limit of a sequence. In Section 3 we prove the existence of the limit.

2 Entropy

The basic notion is here the notion of a partition. We shall introduce it by the help of the following operation $\hat{+}$ on R^2 :

$$(x_1, y_1) \hat{+} (x_2, y_2) = (x_1 + x_2, y_1 + y_2 - 1)$$

Proposition 2.1 $(R^2, \hat{+})$ is a commutative group.

Proof. Evidently $\hat{+}$ is commutative, and it is not difficult to prove that $\hat{+}$ is associative. Further $(0, 1)$ is the neutral element, since $(x_1, y_1) \hat{+} (0, 1) = (x_1 + 0, y_1 + 1 - 1)$. Finally $(x_1, y_1) \hat{+} (-x_1, 2 - y_1) = (0, 1)$.

Remark 2.1 Recall that $(x_1, y_1) \hat{-} (x_2, y_2) = (x_1 - x_2, y_1 - y_2 + 1)$.

Definition 2.2 By a partition of $(1, 0)$ in M we mean a finite collection $\mathcal{A} = \{a_1, \dots, a_n\}$ of elements of M such that

$$a_1 \hat{+} a_2 \hat{+} \dots \hat{+} a_n = (1, 0)$$

The entropy $H(\mathcal{A})$ of the partition \mathcal{A} is defined by the formula

$$H(\mathcal{A}) = \sum_{i=1}^n \varphi(P(a_i))$$

where $\varphi(x) = -x \log x$, if $x \in (0, 1]$, $\varphi(0) = 0$.

Proposition 2.2 If $\mathcal{A} = \{a_1, \dots, a_n\}$ is a partition of $(1, 0)$ and $T\mathcal{A} = \{T(a_1), \dots, T(a_n)\}$, then $T\mathcal{A}$ is a partition of $(1, 0)$, too.

Proof. It is easy to see that $(1, 0) = T((1, 0)) = T(a_1 \hat{+} \dots \hat{+} a_n) = T(a_1) \hat{+} \dots \hat{+} T(a_n)$.

Definition 2.3 If $\mathcal{A} = \{a_1, \dots, a_l\}$, $\mathcal{B} = \{b_1, \dots, b_k\}$ are partitions of $(1, 0)$, then their common refinement is any matrix $\mathcal{S} = \{c_{ij}; i = 1, \dots, k, j = 1, \dots, l\}$ of elements of M such that

$$a_i = c_{i1} \hat{+} \dots \hat{+} c_{il}, i = 1, \dots, k,$$

$$b_j = c_{1j} \hat{+} \dots \hat{+} c_{kj}, j = 1, \dots, l$$

Proposition 2.3 To any partitions there exists their common refinement.

Proof. See [7] Lemma 1, also [2] in a more general situation.

Of course, the common refinement is not defined uniquely. Therefore we consider

$$\mathcal{A} \vee \mathcal{B}$$

as a set of all common refinements of the partitions \mathcal{A}, \mathcal{B} .

The following definition is based on an idea of P. Malický (see [6], [9], [8]).

Definition 2.4 If \mathcal{A} is a partition, then we define $H_n(\mathcal{A}) = \inf \{H(\mathcal{C}); \mathcal{C} \in \mathcal{A} \vee T(\mathcal{A}) \vee \dots \vee T^{n-1}(\mathcal{A})\}$.

The main result of the paper is the following theorem.

Theorem 2.5 There exists $\lim_{n \rightarrow \infty} \frac{1}{n} H_n(\mathcal{A})$.

The proof of the theorem is based on recent results using MV-algebra technique ([7], [2]) and it is presented in the next section.

3 MV-algebra technique

An MV-algebra $M = (M, 0, 1, \neg, \oplus, \odot)$ is a system where \oplus is associative and commutative with neutral element 0, and, in addition, $\neg 0 = 1, \neg 1 = 0, x \oplus 1 = 1, x \odot y = \neg(\neg x \oplus \neg y)$, and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$ for all $x, y \in M$.

MV-algebras stand to the infinite-valued calculus of Lukasiewicz as boolean algebras stand to classical two-valued calculus.

An example of an MV-algebra is the real unit interval $[0, 1]$ equipped with the operations

$$\neg x = 1 - x,$$

$$x \oplus y = \min(1, x + y),$$

$$x \odot y = \max(0, x + y - 1)$$

It is interesting that any MV-algebra has a similar structure. Let G be a lattice-ordered Abelian group (shortly l -group). Let $u \in G$ be a strong unit of G , i.e. for all $g \in G$ there exists an integer $n \geq 1$ such that $nu \geq g$. Let $\Gamma(G, u)$ be the unit interval $[0, u] = \{h \in G; 0 \leq h \leq u\}$ equipped with the operations

$$\neg g = u - g, g \oplus h = u \wedge (g + h), g \odot h = 0 \vee (g + h - u).$$

Then $([0, u], 0, u, \neg, \oplus, \odot)$ is an MV-algebra and by the Mundici theorem ([8], [9]), up to isomorphism, every MV-algebra M can be identified with the unit interval of a unique l -group G with strong unit, $M = \Gamma(G, u)$.

We have seen (Proposition 2.1) that $(R^2, \hat{+})$ is a commutative group. Moreover

Proposition 3.1 $(R^2, \hat{+}, \leq)$ is an l -group, if we consider the partial ordering

$$(x_1, y_1) \leq (x_2, y_2) \iff x_1 \leq x_2, y_1 \geq y_2.$$

Proof. Evidently \leq is a partial ordering, and R^2 is a lattice with respect to this ordering:

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2),$$

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2).$$

Let $(x_1, y_1) \leq (x_2, y_2)$, hence $x_1 \leq x_2, y_1 \geq y_2$.
Then

$$\begin{aligned} (x_1, y_1) \hat{+} (x_3, y_3) &= (x_1 + x_3, y_1 + y_3 - 1) \leq \\ &\leq (x_2 + x_3, y_2 + y_3 - 1) = (x_2, y_2) \hat{+} (x_3, y_3). \end{aligned}$$

Theorem 3.1 $(M, \oplus, \odot, (0, 1), (1, 0))$ is an MV-algebra.

Proof. Consider the l -group $(R^2, \hat{+}, \leq)$. Then

$$M = \{(x, y) \in R^2; (0, 1) \leq (x, y) \leq (1, 0)\}.$$

Moreover,

$$\begin{aligned} ((x_1, y_1) \hat{+} (x_2, y_2)) \wedge (1, 0) &= \\ &= (x_1 + y_1, y_1 + y_2 - 1) \wedge (1, 0) = \\ &= ((x_1 + x_2) \wedge 1, (y_1 + y_2 - 1) \vee 0) \\ &= (x_1 \oplus x_2, y_1 \odot y_2) = \\ &= (x_1, y_1) \oplus (x_2, y_2), \\ ((x_1, y_1) \hat{+} (x_2, y_2) - (1, 0)) \vee (0, 1) &= \\ &= ((x_1 + x_2, y_1 + y_2 - 1) \hat{-} (1, 0) \vee (0, 1) = \\ &= (x_1 + x_2 - 1, y_1 + y_2 - 1 - 0 + 1) \vee (0, 1) = \\ &= ((x_1 + x_2 - 1) \vee 0, (y_1 + y_2) \wedge 1) = \\ &= (x_1 \odot x_2, y_1 \oplus y_2) = \\ &= (x_1, y_1) \odot (x_2, y_2). \end{aligned}$$

Proof of Theorem 2.5.

We shall follow definition 1 in [7]. Let $(a_1, a_2) = (b_1, b_2) \hat{+} (c_1, c_2)$. Then

$$\begin{aligned} T((a_1, a_2)) &= T((b_1, b_2)) \hat{+} T((c_1, c_2)), \\ P((a_1, a_2)) &= P((b_1, b_2)) + P((c_1, c_2)), \\ T((1, 0)) &= (1, 0), P((1, 0)) = 1, \\ P(T((a_1, a_2))) &= P((a_1, a_2)), \end{aligned}$$

hence (M, P, T) is a dynamical system with respect to the MV-algebra $(M, \oplus, \odot, (0, 1), (1, 0))$. Therefore by Theorem 1 of [7] (see also [2] Theorem 4.1) there exists $h(\mathcal{A}, T) = \lim_{n \rightarrow \infty} \frac{1}{n} H_n(\mathcal{A}, T)$.

Example 3.2 Consider $P : M \rightarrow [0, 1]$ defined by $P(x, y) = 1 - y$, and $T : M \rightarrow M$ by the formula $T(x, y) = (2x \pmod{1}, y)$. Then (M, P, T) is a dynamical system.

Remark 3.3 In [7] it is assumed that P is only additive, not necessarily continuous. It would be interesting to describe all additive (not necessarily continuous) probabilities on the Lukasiewicz square.

4 Conclusion

We have shown a formula for computing entropies of dynamical systems on the Lukasiewicz square. It would be interesting to use this formula for the computation of entropy for special cases of entropies as well as different measure preserving transformations. Recall that also some results of [3] are available for such computations. Moreover, it would be interesting to find and use non-continuous probabilities on M and entropies with respect to some measure preserving transformations.

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