Probability theory of fuzzy events

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Abstract

We compare two approaches to probability on systems of fuzzy sets—measures on tribes as introduced by Butnariu and Klement [3] and measures on MV-algebras with product studied by Riečan and Mundici [21]. We show that these two approaches overlap significantly and they confirm the original proposal of probability measure on fuzzy sets by Zadeh [24].

Keywords: Probability measure, fuzzy set, tribe, MV-algebra, MV-algebra with product.

1 Motivation

A lot of knowledge is formulated using vague terms of natural language. Fuzzy sets and fuzzy logic enabled to include such information in computer programs. Fuzzy control succeeded mainly because of this combination of traditional experience and new technology.

In paralel, the development of statistical methods continued. They were mostly based on the classical (Kolmogorovian) probability theory. However, it is quite natural and desirable to combine these two approaches. This could allow an extension of statistical methods to events which are described by fuzzy sets and interpreted as vague statements. This seems to be against the basic paradigm of statistics, because verification/falsification requires exact criteria for the results of experiments. One may object that much experience has been collected in natural sciences, medicine, psychology, sociology, and other "soft sciences" with the use of terms which do not admit exact definitions. In particular, medical diagnosis is still often based on unsharp symptoms (elevated temperature, high blood pressure, overweight, pale skin, etc.). This motivates the effort to introduce fuzzy events to probability theory (or, vice versa, probability to fuzzy set theory). By fuzzy events we mean events whose occurrence is evaluated by more than two truth values. In both approaches presented here, fuzzy events are represented by fuzzy sets which are measurable with respect to some σ-algebra (although also more general elements of MV-algebras will be considered).

2 Different approaches to probability of fuzzy events

Zadeh [24] suggested to define the probability of a fuzzy set f by the formula

$$\mu(f) = \int f \, dP \,, \tag{1}$$

where P is a classical probability measure on some σ-algebra. This is a natural extension of the notion used for crisp sets. However, no justification for exactly this formula was given.

There are many alternative approaches to measure theory on fuzzy sets, see, e.g., [23] and [22], where an extensive bibliography can be found. Here we compare two approaches which – surprisingly – lead from different axioms to similar models.

2.1 Probability on tribes

An axiomatic approach imitating the classical probability theory was suggested by Höhle [10] and developed by Butnariu and Klement [3]. As an analog of a σ-algebra, they suggest a *tribe* of fuzzy sets. Let X be a non-empty set. Following [20], a *tribe* on X is a pentuplet $(\mathcal{T}, \odot, ', 0, \leq)$, where $\mathcal{T} \subseteq [0,1]^X$, $\odot$ is a t-norm, $'$ is a fuzzy negation, 0 is the constant zero function on X, $\leq$ is the fuzzy inclusion, and the following conditions are satisfied:

(T1) $0 \in \mathcal{T}$,

(T2) $f \in \mathcal{T} \implies f' \in \mathcal{T}$,

(T3) $f, g \in \mathcal{T} \implies f \odot g \in \mathcal{T}$,

(T4) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \implies f \in \mathcal{T}$

(the symbol $\nearrow$ denotes monotone increasing convergence). The above definition is a modification of the original notion by Butnariu and Klement. They admitted only the standard negation ($\alpha \mapsto 1-\alpha$) for $'$ and, instead of (T3), (T4), they assumed

(T3+) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \implies \bigodot_{n\in\mathbb{N}} f_n \in \mathcal{T}$

This condition is more general, but the difference is not essential. All results by Butnariu and Klement refer to tribes which satisfy our definition, too.

When there is no risk of confusion, we speak briefly of a tribe $(\mathcal{T}, \odot)$ (as in [2]), resp. of an $\odot$-*tribe* $\mathcal{T}$ (as in [3]). Our notation follows the pattern of [22]. We also speak of an $\odot$-tribe when we need to refer to the t-norm $\odot$, but not to the tribe itself. In particular cases, when $'$ is the standard negation and $\odot$ is the *product t-norm*, $x \odot_{\mathbf{P}} y = x \cdot y$, resp. the *Łukasiewicz t-norm*, $x \odot_{\mathbf{L}} y = \max(0, x+y-1)$, we speak of a *product tribe*, resp. a *Łukasiewicz tribe.*

A *probability measure* on a tribe $(\mathcal{T}, \odot, ', 0, \leq)$ is a functional $\mu\colon \mathcal{T} \to [0,1]$ such that

(M1) $\mu(0) = 0$, $\mu(1) = 1$,

(M2) $f, g \in \mathcal{T} \implies \mu(f \odot g) + \mu(f \oplus g) = \mu(f) + \mu(g)$, where $\oplus$ is the t-conorm dual to $\odot$,

(M3) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \implies \mu(f_n) \to \mu(f)$,

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Conditions (M3), (M4) represent the *σ-order continuity* of μ. In preceding studies, mainly [3], condition (M4) was not required. This resulted from analogy with σ-algebras where (M3), (M4) are equivalent. However, without (M4) we obtain so-called *support measures* of the form

$$\mu(f) = P(\operatorname{supp} f), \qquad f \in \mathcal{T},$$

where $\operatorname{supp} f = \{x \in X \mid f(x) > 0\}$ is the *support* of a fuzzy set f and P is a classical measure. Such measures depend only on the support and do not distinguish among positive membership degrees. This seems to be hardly motivated by applications.

Frank t-norms $\odot^{\mathbf{F}}_{\lambda}$, $\lambda \in [0,\infty]$, play an important role in the characterization of probability measures on tribes. They are defined by (see [12, Example 2.9.3])

$$x \odot^{\mathbf{F}}_{\lambda} y = \log_{\lambda}\left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right)$$

if $\lambda \in\,]0,\infty[\setminus \{1\}$, $\odot^{\mathbf{F}}_0 = \odot_{\mathbf{M}} = \min$, $\odot^{\mathbf{F}}_1 = \odot_{\mathbf{P}}$, and $\odot^{\mathbf{F}}_{\infty} = \odot_{\mathbf{L}}$. Frank t-norms are Archimedean for $\lambda > 0$ and strict for $\lambda \in\,]0,\infty[$.

It has been found that all (σ-order continuous) probability measures on a $\odot$-tribe are of the form (1) if $'$ is the standard negation and $\odot$ is an Archimedean Frank t-norm (in particular, the product or the Łukasiewicz t-norm). On the other hand, these are the only Archimedean t-norms for which formula (1) defines a measure on a non-Boolean tribe. Moreover, only those Archimedean t-norms which are in some sense equivalent to Frank t-norms admit (σ-order continuous) probability measures at all.

2.2 Probability on MV-algebras

Another approach to measures of fuzzy sets is based on MV-algebras [6].

A *state*[1] on a σ-complete MV-algebra $\mathcal{T}$ is a map-

[1] Usually a state is a synonym for a probability measure. Here we use these notions in different contexts and we distinguish them. However, we shall see that they coincide in important cases.

ping $\mu\colon \mathcal{T} \to [0,1]$ satisfying the following conditions:

(S1) $\mu(1) = 1$,

(S2) $f, g \in \mathcal{T},\ f \odot_L g = 0 \implies \mu(f \oplus_L g) = \mu(f) + \mu(g)$,

(S3) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \implies \mu(f_n) \to \mu(f)$.

The state space is a convex set; its extreme points are called *pure states*.

Due to properties of Łukasiewicz operations, the conjunction of (S1), (S2) is equivalent to the conjunction of (M1), (M2) (for strict t-norms, (S2) is much weaker than (M2)). Condition (S3) is identical to (M3) and it implies also (M4) for σ-complete MV-algebras.

Łukasiewicz tribes form a special class of MV-algebras. They are characterized in [7] as those σ-complete MV-algebras which admit a *separating set* of pure states, i.e., for each $a, b \in \mathcal{T}$, $a \neq b$, there is a pure state s such that $s(a) \neq s(b)$. Although not all σ-complete MV-algebras satisfy this property, it is reasonable to require it in probability theory. This has a Boolean analogy: Among general σ-complete Boolean algebras, only σ-algebras of subsets of a set are usually considered a good basis of a probability theory. The relation of Łukasiewicz tribes to σ-complete MV-algebras is the same as that of σ-algebras to general σ-complete Boolean algebras.

As a consequence, the notions of state and probability measure, as introduced here, coincide for Łukasiewicz tribes. We shall see that this happens in a much more general context.

One important feature of MV-algebras (not achieved in other fuzzifications of Boolean algebras) is the existence of partitions of unity and their joint refinements. However, the coarsest joint refinement of two partitions of unity need not exist and there is no canonical formula for such refinements (see [14] for more details). Such a formula is highly desirable for probability theory. Therefore the basics of statistics (central limit theorem, etc.) were developed in [21] only for the case when $\mathcal{T}$ is an *MV-algebra with product*, i.e., a pair $(\mathcal{T}, \cdot)$, where $\mathcal{T}$ is an MV-algebra and $\cdot$ is a commutative and associative binary operation on $\mathcal{T}$ satisfying:

(P1) $1 \cdot a = a$,

(P2) $a \cdot (b \odot_L c') = (a \cdot b) \odot_L (a \cdot c)'$.

For the standard MV-algebra, $[0,1]$ with the Łukasiewicz operations, the algebraic product is the only operation making it an MV-algebra with product [21]. More generally, a Łukasiewicz tribe forms an MV-algebra with product iff it is equipped with the algebraic product as $\cdot$; then we call it a *Łukasiewicz tribe with product*.

3 Comparison

The above two approaches coincide in the following important case:

3.1 Theorem
Let $\odot^{\mathbf{F}}_{\lambda}$, $\lambda \in\ ]0,\infty[$, be a strict Frank t-norm. Then $\odot^{\mathbf{F}}_{\lambda}$-tribes with the standard negation as $'$ are exactly Łukasiewicz tribes with product (when equipped with the respective operations); states on them coincide with probability measures.

Proof: Equality of notions of tribe: Due to [3], every $\odot^{\mathbf{F}}_{\lambda}$-tribe is also a Łukasiewicz tribe. In particular, every product tribe is a Łukasiewicz tribe with product (the products $\cdot, \odot_{\mathbf{P}}$ coincide). According to [18], every product tribe is also a tribe with respect to any measurable t-norm, in particular, a $\odot^{\mathbf{F}}_{\lambda}$-tribe. The result of [18] has been generalized to all strict Frank t-norms in [15]: Every $\odot^{\mathbf{F}}_{\lambda}$-tribe is a tribe with respect to any measurable t-norm, in particular, a product tribe (and also a Łukasiewicz tribe with product).

Equality of notions of state and probability measure: Due to [3], each probability measure on a $\odot^{\mathbf{F}}_{\lambda}$-tribe is also a state on the corresponding Łukasiewicz tribe. For the second implication, suppose that μ is a state on a Łukasiewicz tribe with product. For any two elements f, g, the quadruplet $f \odot_{\mathbf{P}} g, f \odot_{\mathbf{P}} g', f' \odot_{\mathbf{P}} g, f' \odot_{\mathbf{P}} g'$ forms a partition of unity, thus

$$\mu(f \odot_{\mathbf{P}} g)+\mu(f \odot_{\mathbf{P}} g')+\mu(f' \odot_{\mathbf{P}} g)+\mu(f' \odot_{\mathbf{P}} g') = 1,$$

$$\begin{aligned}\mu(f \oplus_{\mathbf{P}} g) &= \mu((f' \odot_{\mathbf{P}} g')') = 1 - \mu(f' \odot_{\mathbf{P}} g')\,,\\ \mu(f) &= \mu(f \odot_{\mathbf{P}} g) + \mu(f \odot_{\mathbf{P}} g')\,,\\ \mu(g) &= \mu(f \odot_{\mathbf{P}} g) + \mu(f' \odot_{\mathbf{P}} g)\,.\end{aligned}$$

This implies

$$\begin{aligned}\mu(f \oplus_{\mathbf{P}} g) &= \mu(f \odot_{\mathbf{P}} g) + \mu(f \odot_{\mathbf{P}} g') + \mu(f' \odot_{\mathbf{P}} g)\\ &= \big(\mu(f \odot_{\mathbf{P}} g) + \mu(f \odot_{\mathbf{P}} g')\big)\\ &\quad + \big(\mu(f' \odot_{\mathbf{P}} g) + \mu(f \odot_{\mathbf{P}} g)\big)\\ &\quad - \mu(f \odot_{\mathbf{P}} g)\\ &= \mu(f) + \mu(g) - \mu(f \odot_{\mathbf{P}} g)\,.\end{aligned}$$

We have proved that μ is a probability measure on the product tribe.

To prove that μ is also a probability measure on the $\odot^{\mathbf{F}}_{\lambda}$-tribe, we need the integral representation of μ in the form (1) (see [17,20] for details of this highly non-trivial proof). According to [9], Frank t-norm $\odot^{\mathbf{F}}_{\lambda}$ and its dual t-conorm $\oplus^{\mathbf{F}}_{\lambda}$ possess the following property for all $\alpha, \beta \in [0,1]$:

$$\alpha \odot^{\mathbf{F}}_{\lambda} \beta + \alpha \oplus^{\mathbf{F}}_{\lambda} \beta = \alpha + \beta\,.$$

Due to the additivity of integrals,

$$\begin{aligned}\mu(f \oplus^{\mathbf{F}}_{\lambda} g) &= \int (f \oplus^{\mathbf{F}}_{\lambda} g)\, dP\\ &= \int (f + g - (f \odot^{\mathbf{F}}_{\lambda} g))\, dP\\ &= \int f\, dP + \int g\, dP - \int (f \odot^{\mathbf{F}}_{\lambda} g)\, dP\\ &= \mu(f) + \mu(g) - \mu(f \odot^{\mathbf{F}}_{\lambda} g)\,.\end{aligned}$$

In fact, the integral representation implies the equivalence of all versions of states and probability measures studied here. However, it is based on advanced proofs; we used simpler arguments where possible. □

3.2 COROLLARY
Łukasiewicz tribes with product are exactly product tribes; states on them coincide with probability measures.

It is surprising is that we obtain the same notion from two axiomatic systems where condition (S2) (formulated for the *Łukasiewicz* t-norm) is much different from condition (M2) (formulated for the *product* t-norm, resp. a *strict Frank* t-norm).

4 Conclusions

The overlapping of MV-algebras with product and tribes (w.r.t. strict Frank t-norms) shows that two approaches to probability on fuzzy sets converge to essentially the same notions. This opens a new field for further investigations, because some results can be directly translated from one context to the other. Among others, generalizations of the central limit theorem, laws of large numbers [21], and results about entropy can be applied to product (and some other) tribes, too. This applies also to recent studies of conditional probability on MV-algebras with product [11,13] which offer a solution to an open problem from [21].

On the other hand, a lot of results were derived for tribes, e.g., decomposition theorems, extensions of Lyapunov theorem [1], and applications to games with fuzzy coalitions [3]. These can be applied to MV-algebras with product (at least in the case when they are Łukasiewicz tribes).

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