

Reasoning with uncertainty and context-dependent languages

Claudio Sossai

ISIB-CNR, Corso Stati Uniti 4, 35127 Padova, Italy

sossai@isib.cnr.it

Abstract

A topos of presheaves can be seen as an extension of classical set theory, where sets vary over informational states, therefore it is a powerful and expressive mathematical framework. I introduce a suitable topos of presheaves where imprecise probabilities and imprecise probabilistic reasoning can be represented. In this way we obtain a mathematical definition of imprecise-probabilistic sets. A valid and complete proof system, w.r.t. the intended semantics of imprecise probabilities, is described using the internal language of the topos. **Key words:** reasoning with uncertainty, topos theory, sheaf theory

1 Introduction

It is a well known fact that topos and in particular Sheaves and presheaves theory describe a set theory whose objects vary over informational states. From the logical point of view, the novelty stays in the local aspect of truth, i.e. the question is not whether a proposition is true (or false) but where, in which informational state or context, a proposition is true (or false). For this reason, the internal language of a topos can be seen as a context-dependent language. In this work I proceed in the investigation of a claim presented in a previous work (see [10]): key aspects of uncertainty and hence of uncertainty reasoning can be mathematically described using context-dependent logical languages. For this reason I believe that topos theory is a powerful mathematical tool to describe uncertainty in its various aspects: linguistic uncertainty (possibility theory), stochastic

uncertainty (probability theory) and their generalizations: evidence theory and imprecise probabilities (see e.g. [5, 8, 9]). In possibility theory, a possibility distribution can be seen as a context where a linguistic construct takes meaning. For example, a particular possibility distribution that describes the linguistic term *long* can be seen as a context where *long* takes meaning. Using this simple observation, I have proved that the internal language of the topos of presheaves built over the category whose objects are sets of possibility distributions, contains a valid and complete logic for possibilistic reasoning. In this work I prove that all the above observations and mathematical hypotheses hold if we extend linguistic uncertainty by adding stochastic uncertainty, providing new evidence to the claim that a key aspect of uncertainty is the contextualization of truth and hence of the entire language. To this aim, I have investigated the case of imprecise probabilities, i.e. functions $P^* : \mathcal{B} \rightarrow [0, 1]$ defined as: $P^*(L) = \bigvee \{ \mu(L) \mid \mu \in P \}$ (where P is a set of probability measures and \mathcal{B} is a Boolean algebra). Belief functions are special cases of imprecise probabilities, in fact they can be seen as the lower envelope of the probability measures that dominate them. Possibilities are special cases of belief functions, in fact a possibility measure is a belief function whose focal elements are nested. Therefore imprecise probabilities can be seen as a framework where both linguistic and stochastic uncertainty can be represented. In this work I prove that the same mathematical construction used to define the process of contextualizing linguistic uncertainty is sufficient to describe also stochastic uncertainty. More precisely, I will con-

struct the topos of presheaves that vary over the category of contexts (in this case, sets of sets of probability measures) and I will prove that a valid and complete logic for imprecise probabilistic reasoning (see [7]) can be represented in the internal language.

2 U-Sets as imprecise-probabilistic sets

Let \mathcal{B} be a finite Boolean algebra and \mathcal{P} be the set of all probability measures μ defined on \mathcal{B} .

Def. 1 A context is an element of $\mathbf{U} = 2^{2^{\mathcal{P}}}$, different from \emptyset and $\{\emptyset\}$, where $2^{\mathcal{P}}$ is the power set of \mathcal{P}

It is easy to see that \mathbf{U} determines a category in the following way: the objects are the elements of \mathbf{U} , the arrows are the immersions, i.e. there is an arrow $f : B \rightarrow A$ iff $B \subseteq A$. Again, I will call this category \mathbf{U} .

Def. 2 \mathbf{U} -Sets is the topos of presheaves over \mathbf{U}

The internal language of \mathbf{U} -Sets is defined as usual (see [2]). To fix the notation I will only recall the definition of the connectives. Formulae are terms of type Ω , if $F \xrightarrow{\alpha} \Omega$, $F \xrightarrow{\beta} \Omega$ are formulae, then the connectives are defined, by induction, for all A , $a \in F(A)$ as: $(\alpha \wedge \beta)_A(a) = (\alpha)_A(a) \cap (\beta)_A(a)$; $(\alpha \vee \beta)_A(a) = (\alpha)_A(a) \cup (\beta)_A(a)$; $(\alpha \rightarrow \beta)_A(a) = \{f \in \max^A : \Omega(f)((\alpha)_A(a)) \subseteq \Omega(f)((\beta)_A(a))\}$; $(\neg \alpha)_A(a) = \{f \in \max^A : \Omega(f)((\alpha)_A(a)) = \emptyset\}$, where \max^A is the set of all arrows with codomain A . If $F \times G \xrightarrow{\alpha} \Omega$, i.e. α contains an additional variable x_G of type G , then: $((\exists x_G)\alpha)_A(a) = \{f \in \max^A : (\exists b \in G(B))((\alpha)_B(\langle F(f)(a), b \rangle)) = \max^B\}$; $((\forall x_G)\alpha)_A(a) = \{f \in \max^A : (\forall C \xrightarrow{g} B)(\forall b \in G(C))((\alpha)_C(\langle F(f \circ g)(a), b \rangle)) = \max^C\}$. The *semantical entailment* is defined, for $F \xrightarrow{\alpha} \Omega$, $F \xrightarrow{\beta} \Omega$, as: $\alpha \models \beta$ iff $(\forall A)(\forall a \in F(A))(\alpha_A(a) \subseteq \beta_A(a))$. Moreover, validity of a formula α in \mathbf{U} -Sets is defined as: $\models \alpha$ iff $(\forall A)(\forall a \in F(A))(\alpha_A(a) = \max^A)$. As usual in topos theory:

Def. 3 Imprecise-probabilistic sets are objects of \mathbf{U} -Sets of power type.

3 Reasoning about upper probabilities

Let me recall a set of axioms for imprecise probabilities. The axiomatization is based on the notion of set cover. A set S is covered n times by a multiset $\{\{S_1, \dots, S_m\}\}$ of sets if every element in S appears in at least n sets from S_1, \dots, S_m : for all $x \in S$ there exists i_1, \dots, i_n in $\{1, \dots, m\}$ s.t. $x \in S_{i_j}$ for all $j \leq n$. An (n, k) -cover of (L, Ω) is a multiset $\{\{L_1, \dots, L_m\}\}$ that covers Ω k times and covers L $n+k$ times. I will write $\bigvee_X \mu(L)$ for $\bigvee\{\mu(L) \mid \mu \in X\}$. The following holds (see [1]):

Theorem 1 Suppose that Ω is a set, Σ is an algebra of subsets of Ω and $P^* : \Sigma \rightarrow R$. Then there exists a set X of probability measures with $P^* = \bigvee_X \mu$ iff P^* satisfies the following properties:

1. $P^*(\emptyset) = 0$
2. $P^*(\Omega) = 1$
3. for all natural numbers m, n, k and all subset L_1, \dots, L_m in Σ if $\{\{L_1, \dots, L_m\}\}$ is an (n, k) -cover of (L, Ω) , then $k + nP^*(L) \leq \sum_{i \leq m} P^*(L_i)$.

It can be proved (see [7]) that if Ω is finite then there exists a finite integer number $n_{|\Omega|}$ s.t. the above theorem holds with $m, n, k \leq n_{|\Omega|}$, i.e. property 3 in the theorem must be checked only for a finite number of m, n, k .

Let me describe some useful objects of \mathbf{U} -Sets.

Def. 4 Δ^S (where S is a set) is the constant functor defined as $\Delta^S(A) = S$ for every object A and $\Delta^S(f) = 1_S$ for every arrow f .

In particular, I will consider $\Delta^{\mathcal{B}}$, where \mathcal{B} is the Boolean algebra and Δ^R , where R is the set of real numbers. Using the internal language of \mathbf{U} -Sets, it is possible to prove that the elements of the constant presheaf Δ^R satisfy the properties of Dedekind cuts (see [4]), hence I will take Δ^R as the set of real numbers in \mathbf{U} -Sets. To describe upper measures in \mathbf{U} -Sets, let me introduce the following functor:

Def. 5 Let $\mathcal{S} : \mathbf{U}^{op} \rightarrow \mathbf{Sets}$ be defined as:

1. $\mathcal{S}(A) = \{k : A \rightarrow R\}$, i.e. the set of all functions k from A to the set R of usual real numbers
2. for $B \xrightarrow{f} A$, $\mathcal{S}(A) \xrightarrow{\mathcal{S}(f)} \mathcal{S}(B)$ is the function that sends every $g \in \mathcal{S}(A)$ to g restricted to B .

Note that Δ^R can be embedded in \mathcal{S} . In fact, for every usual real number $r \in R$, let $\lceil r \rceil \in \mathcal{S}(A)$ be the function that sends A to r , i.e. $\lceil r \rceil : A \mapsto r$, then $\iota^{\Delta^R} : \Delta^R \rightarrow \mathcal{S}$ is defined, for all $A, r \in \Delta^R(A)$ as : $\iota_A^{\Delta^R}(r) = \lceil r \rceil$. When the context is clear, to simplify the notation, I will write r for the internal representation of the real number r , i.e. $r : 1 \rightarrow \mathcal{S}$ defined for all A as $r_A(\star) = \lceil r \rceil$. Now we have a mathematical machinery sufficient to give the categorical definition of upper measure.

Def. 6 For every $L \in \Delta^{\mathcal{B}}$ let $p_A^*(L) \in \mathcal{S}(A)$ be the function that maps every $X \in A$ to $\bigvee_X \mu(L)$.

The variable upper measure can be represented as follows:

Def. 7 For every A , P_A^* is the function $P_A^* : \Delta^{\mathcal{B}} \rightarrow \mathcal{S}(A)$ that sends $L \in \Delta^{\mathcal{B}}$ to the function $p_A^*(L) \in \mathcal{S}(A)$.

It is easy to see that the family $\{P_A^* : A \text{ object of } U\}$ defines a natural transformation $P^* : \Delta^{\mathcal{B}} \rightarrow \mathcal{S}$. Every element L of the Boolean algebra \mathcal{B} is represented by the arrow $L : 1 \rightarrow \Delta^{\mathcal{B}}$ defined as $L_A(\star) = L \in \Delta^{\mathcal{B}}(A)$. The following lemma shows that P^* is the internal representation of upper measures.

Lemma 1 For all A , $P^*(L) = r$ is true in A (i.e. $(P^*(L) = r)_A \langle \star, \star \rangle = \max^A$) iff $(\forall X \in A)(\bigvee_X \mu(L) = r)$

Proof As usual, equality $\mathcal{S} \times \mathcal{S} \xrightarrow{\cong} \Omega$ is defined as the characteristic arrow of the diagonal $\mathcal{S} \xrightarrow{\delta} \mathcal{S} \times \mathcal{S}$. It holds that: $(P^*(L) = r)_A \langle \star, \star \rangle = \{B \xrightarrow{f} A \mid \langle \mathcal{S}(f)((P^*(L))_A(\star)), \mathcal{S}(f)(r_A(\star)) \rangle \in \delta_B(\mathcal{S}(B))\} = \{B \xrightarrow{f} A \mid \mathcal{S}(f)((P^*(L))_A(\star)) = \mathcal{S}(f)(r_A(\star))\} = \{B \xrightarrow{f} A \mid (\forall X \in B)(\bigvee_X \mu(L) = r)\}$. Using the above equalities it easy to see that $(P^*(L) = r)_A \langle \star, \star \rangle = \max^A$ iff $(\forall X \in A)(\bigvee_X \mu(L) = r)$. \square

Due to the fact that $1 \times 1 \simeq 1$, to simplify the notation, I will assume that all arrows with domain $1 \times .. \times 1$ are defined on 1 implicitly assuming the above morphism. The usual order relation over R can be defined in the internal language as follows:

Def. 8 For every A , $\langle h, k \rangle \in \mathcal{S}(A) \times \mathcal{S}(A)$, let $\geq_A : \mathcal{S}(A) \times \mathcal{S}(A) \rightarrow \Omega(A)$ be the function defined as $\geq_A(\langle h, k \rangle) = \{B \xrightarrow{f} A : (\forall X \in B)(\mathcal{S}(f)(h)(X) \geq \mathcal{S}(f)(k)(X))\}$.

Lemma 2 The family of functions $\{\geq_A : A \text{ object of } U\}$ defines a natural transformation.

Proof We must prove that for every arrow $B \xrightarrow{f} A$ it holds that $\geq_B \circ \langle \mathcal{S}(f), \mathcal{S}(f) \rangle = \Omega(f) \circ \geq_A$. Let $\langle h, k \rangle \in \mathcal{S}(A) \times \mathcal{S}(A)$. Note that:

1. $(\Omega(f) \circ \geq_A)(\langle h, k \rangle) = \{C \xrightarrow{g} B : f \circ g \in \geq_A(\langle h, k \rangle)\} = \{C \xrightarrow{g} B : (\forall X \in C)(\mathcal{S}(f \circ g)(h)(X) \geq \mathcal{S}(f \circ g)(k)(X))\}$.
2. $\geq_B(\langle \mathcal{S}(f)(h), \mathcal{S}(f)(k) \rangle) = \{C \xrightarrow{g} B : (\forall X \in C)((\mathcal{S}(g) \circ \mathcal{S}(f))(h)(X) \geq (\mathcal{S}(g) \circ \mathcal{S}(f))(k)(X))\}$

The identity follows from the fact that $\mathcal{S}(f \circ g) = \mathcal{S}(g) \circ \mathcal{S}(f)$ \square

Let me note that the usual operations of product and sum defined over R can be extended to \mathcal{S} . In fact, let the sum be the natural transformation $+ : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ defined for $\langle g, k \rangle \in \mathcal{S}(A) \times \mathcal{S}(A)$ and $X \in A$ as $+(\langle g, k \rangle)(X) = g(X) + k(X) \in R$. To simplify the notation I will often write $g + k$ for $+(\langle g, k \rangle)$. Product is defined similarly, and the reader can easily check the naturality of the two transformations. Using the above definitions I will prove that the internal language $\mathcal{L}^{\mathbf{U}\text{-Sets}}$ of $\mathbf{U}\text{-Sets}$ contains a sublanguage \mathcal{L}^{P^*} , that is a conservative extension of \mathcal{L}^{QU} (see [7]) a valid and complete logic w.r.t. the semantics of upper probabilities. An atomic formula of the language \mathcal{L}^{QU} is a term of the form $r_1 l(L_1) + \dots + r_n l(L_n) \geq r$, where $L_i \in \mathcal{B}$ and $r, r_i \in R$. A formula is a Boolean combination of atomic formulae. A model is a set of measures X and the definition of semantics is:

- $X \models r_1 l(L_1) + \dots + r_n l(L_n) \geq r$ iff $r_1 \bigvee_X \mu(L_1) + \dots + r_n \bigvee_X \mu(L_n) \geq r$
- $M \models \neg_{\mathcal{B}} f$ iff $M \not\models f$
- $M \models f \wedge_{\mathcal{B}} g$ iff $M \models f$ and $M \models g$

where $\neg_{\mathcal{B}}$ and $\wedge_{\mathcal{B}}$ are the Boolean connectives. When the context is clear I will drop the subscript.

Def. 9 The translation $T : \mathcal{L}^{QU} \rightarrow \mathcal{L}^{U\text{Sets}}$ is defined as:

- $T(r_1 l(L_1) + \dots + r_n l(L_n) \geq r) = r_1 P^*(L_1) + \dots + r_n P^*(L_n) \geq r$
- $T(\neg_{\mathcal{B}} f) = \neg T(f)$
- $T(f \wedge_{\mathcal{B}} g) = T(f) \wedge T(g)$

Let me check that definition 9 is a good definition, i.e. if $f = g$ then $T(f) = T(g)$. To this aim we need the following lemma.

Lemma 3 For all sets of measures X it holds that: $(T(f))_{\{X\}}(\star) = \max^{\{X\}}$ iff $X \models f$.

Proof By induction on the length of f

- let $f = \sum_{i \leq n} r_i l(L_i) \geq r$, it is easy to see that $(T(f))_{\{X\}}(\star) = \max^{\{X\}}$ iff $\sum_{i \leq n} r_i \bigvee_X \mu(L_i) \geq r$ iff $X \models f$
- let $f = f_1 \wedge f_2$ then $\max^{\{X\}} = (T(f_1 \wedge f_2))_{\{X\}}(\star)$ iff $(T(f_1))_{\{X\}}(\star) = \max^{\{X\}}$ and $(T(f_2))_{\{X\}}(\star) = \max^{\{X\}}$ iff (by inductive hypothesis) $X \models f_1$ and $X \models f_2$ iff $X \models f$
- Let $f = \neg f_1$.
 - if $\max^{\{X\}} = (T(\neg f_1))_{\{X\}}(\star) = (\neg T(f_1))_{\{X\}}(\star)$ then not $\max^{\{X\}} = (T(f_1))_{\{X\}}(\star)$, therefore, by inductive hypothesis, not $X \models f_1$ and hence $X \models \neg f_1$.
 - if $X \models \neg f_1$ then not $X \models f_1$ and hence not $\max^{\{X\}} = (T(f_1))_{\{X\}}(\star)$. Note that $\max^{\{X\}} = \{1_{\{X\}} : \{X\} \rightarrow \{X\}\}$, hence $(T(f_1))_{\{X\}}(\star) = \emptyset$. From this we obtain: $(T(\neg f_1))_{\{X\}}(\star) = (\neg T(f_1))_{\{X\}}(\star) = \{B \xrightarrow{f} \{X\} \mid \Omega(f)((T(f_1))_{\{X\}}(\star)) = \emptyset\} = \{B \xrightarrow{f} \{X\} \mid \Omega(f)(\emptyset) = \emptyset\} = \max^{\{X\}}$.

□

Note that from lemma 3 we have that $T(f) = \text{true}$ iff $(\forall X)(X \models f)$ and that $T(f) = \text{false}$ iff $(\forall X)(X \not\models f)$ therefore if f is s.t. $(\exists X)(X \not\models f)$ and $(\exists Y)(Y \models f)$ then $T(f)$ is different from true and different from false .

Fact 1 If $f = g$ then $T(f) = T(g)$

Proof

If $f = g$ then $(\forall X)(X \models f$ iff $X \models g)$ iff (by lemma 3) $(\forall X)(T(f))_{\{X\}}(\star) = T(g)_{\{X\}}(\star)$. From this it holds that for all A $(T(f))_A(\star) = (T(g))_A(\star)$ therefore $T(f) = T(g)$. □

Note that as a consequence of fact 1 we have that $\neg \neg T(f) = \neg(T(\neg f)) = T(\neg \neg f) = T(f)$ therefore the formulae of \mathcal{L}^{P^*} behave classically. A valid and complete proof system is given by the following set of axioms (AX^{UP}) and modus ponens (see [7]):

- TAUT: all instances of propositional tautologies in \mathcal{L}^{QU}
- MP: from f and $f \rightarrow g$ infer g
- INEQ: all instances of valid formulae about linear inequalities
- L1: $l(\perp_{\mathcal{B}}) = 0$
- L2: $l(\top_{\mathcal{B}}) = 1$
- L3: $l(L) \geq 0$
- L4: $l(L_1) + \dots + l(L_m) - nl(L) \geq k$ if $L \rightarrow \bigvee_{J \subseteq \{1, \dots, m\}, |J|=k+n} \bigwedge_{j \in J} L_j$ and $\bigvee_{J \subseteq \{1, \dots, m\}, |J|=k} \bigwedge_{j \in J} L_j$ are propositional tautologies.

An inequality formula is a formula of the form $r_1 x_1 + \dots + r_n x_n \geq r$. It is said valid if it is true under every possible assignment of real numbers to variables. To get an instance of INEQ, we replace each variable x_i that occurs in a valid inequality formula by a primitive likelihood term of the form $l(L_i)$. L1, L2 and L4 are the defining properties of upper probabilities. Validity and completeness means that for all X it holds that $X \models f$ iff f is provable using the above sets of axioms and rules.

Corollary 1 For all formula $f \in \mathcal{L}^{QU}$ it holds that $T(f) = true$ iff $(\forall X)(X \models f)$

Proof $T(f) = true$ iff $(\forall A)((T(f))_A(\star) = max^A$ iff $(\forall X \in A)((T(f))_{\{X\}}(\star) = max^{\{X\}})$ iff $(\forall X)(X \models f)$. \square

Due to the fact that every axiom of AX^{up} is true in every model we have that:

Theorem 2 All the formulae of \mathbf{U} -Sets obtained as translations of the axioms AX^{up} are true in \mathbf{U} -Sets

Therefore I can assume as axioms the set $AX^U = \{T(f) | f \in AX^{up}\}$. Due to the fact that the proof system of the internal language of a topos is valid, i.e. from *true* formulae we derive *true* formulae, we have that if $T(g)$ can be proved from the set of axioms AX^U then $T(g)$ is true. This proves validity:

Theorem 3 If $T(f)$ is provable in \mathbf{U} -Sets using AX^U as proper axioms, then $(\forall X)(X \models f)$.

As I have noticed the formulae of \mathcal{L}^{P^*} behave classically as the formulae of \mathcal{L}^{QU} . Moreover, all the negated atomic formulae of \mathcal{L}^{QU} can be written as positive inequality (i.e. $\neg(l(L) \geq s)$ can be written as $\neg(l(L) > -s)$). This property holds also in \mathbf{U} -Sets:

Fact 2 $\neg(P^*(L) \geq r) = P^*(L) < r = \neg P^*(L) > -r$

Proof For all A it holds that: $\neg(P^*(L) \geq r)_A(\star) = \{B \xrightarrow{f} A | \Omega(f)(P^*(L) \geq r)_A(\star) = \emptyset\} = \{B \xrightarrow{f} A | \neg(\exists C \xrightarrow{g} B)(f \circ g \in (P^*(L) \geq r)_A(\star))\} = \{B \xrightarrow{f} A | (\forall X \in B)(\neg(\bigvee_X \mu(L) \geq r))\} = \{B \xrightarrow{f} A | (\forall X \in B)(\bigvee_X \mu(L) < r)\} = (P^*(L) < r)_A(\star)$. The second equality follows easily. \square

For completeness it remains to prove that the proof system of \mathbf{U} -Sets restricted to \mathcal{L}^{P^*} , the part of the internal language that represents \mathcal{L}^{QU} , is as strong as the proof system of \mathcal{L}^{QU} . I have proved that all the axioms are *true* it remains to prove that MP can be represented as a valid rule. The proof of a formula $\vdash \beta$ from $\vdash \alpha$ and $\vdash \alpha \rightarrow \beta$ is an easy exercise in intuitionistic sequent calculus. What remains to see is that the translation is

faithful w.r.t. implication. This is a consequence of the fact that the formulae of \mathcal{L}^{P^*} behave classically but it can also be checked directly:

Fact 3 $T(f \rightarrow g) = T(\neg(f \wedge \neg g)) = \neg(T(f) \wedge \neg(T(g))) = T(f) \rightarrow T(g)$.

Proof For all A it holds by definition that $(T(f) \rightarrow T(g))_A(\star) = \{f \in max^A | \Omega(f)((T(f))_A(\star)) \subseteq \Omega(f)((T(g))_A(\star))\}$ while $(\neg(T(f) \wedge \neg T(g)))_A(\star) = \{f \in max^A | \Omega(f)((T(f))_A(\star) \cap (T(g))_A(\star)) = \emptyset\}$. From this using the fact that if $S_1, S_2 \in \Omega(A)$ then $\Omega(f)(S_1 \cap S_2) = \Omega(f)(S_1) \cap \Omega(f)(S_2)$ we obtain that $(\neg(T(f) \wedge \neg T(g)))_A(\star) = \{f \in max^A | \Omega(f)((T(f))_A(\star) \cap \Omega(f)((\neg T(g))_A(\star)) = \emptyset\}$. Note that $\Omega(f)((\neg T(g))_A(\star)) = \{k \in max^B | f \circ k \notin (T(g))_A(\star)\}$, from this the fact follows. \square

By fact 3 and from the fact that the rule *MP* is provable in intuitionistic sequent calculus, we obtain that the translation preserves also the (only) deductive rule of \mathcal{L}^{QU} . Assume that $(\forall X)(X \models f)$. Then by completeness of \mathcal{L}^{QU} we have only two possibilities: f is an axiom and therefore $T(f)$ is provable in \mathbf{U} -Sets or it is obtained using *MP* and also in this case $T(f)$ is provable in \mathbf{U} -Sets. Therefore we obtain completeness for the translation. There is also a way of reproducing the same argument used in [7] to prove completeness from inside \mathbf{U} -Sets. This proof is interesting because it defines an effective algorithm to decide whether a formula is satisfiable i.e. if $T(f) \neq false$. As a consequence of Lemma 3 we have that

Lemma 4 $(\exists X)(X \models f)$ iff $T(f) \neq false$.

Proof

- Assume that $(\exists X)(X \models f)$ then by lemma 3 it holds that $T(f) \neq false$.
- If $T(f) \neq false$ then $(\exists A)((T(f))_A(\star) = max^A)$. Therefore $(\forall X \in A)(\forall g : \{X\} \rightarrow A)(\Omega(g)((T(f))_A(\star)) = max^{\{X\}})$. By naturality of $T(f)$ it holds that: $(T(f))_{\{X\}}(\star) = \Omega(g)(T(f)_A(\star)) = max^{\{X\}}$. From this applying lemma 3 we obtain $X \models f$.

\square

Using lemma 4 it is possible to define an algorithm to check whether $T(f) \neq false$ which will be the key for completeness. For every f write f in normal form $f = \bigvee_{j \in J} (\bigwedge_{i_j \in I_j} g_{i_j})$. $X \models f$ iff there is $j \in J$ s.t. $X \models \bigwedge_{i_j \in I_j} g_{i_j}$ therefore we can assume that $f = \bigwedge_{i \in I} g_i$. Let $\mathcal{B} = \{\rho_1, \dots, \rho_N\}$ where $\rho_1 = \top$ and $\rho_N = \perp$. Therefore f is equivalent to a formula f' where each positive conjunct g_i is of the form $r_{i,1}l(\rho_1) + \dots + r_{i,N}l(\rho_N) \geq s_i$, while a negative conjunct has the form $r_{i,1}l(\rho_1) + \dots + r_{i,N}l(\rho_N) < s_i$ that is equivalent to $(-r_{i,1})l(\rho_1) + \dots + (-r_{i,N})l(\rho_N) > -s_i$. Replacing each $l(\rho_j)$ by a variable x_j we obtain a system of linear inequalities \bar{f} . If f is satisfiable in some upper probability structure X then we can take $x_j = \bigvee_X \mu(\rho_j)$. This gives a solution for \bar{f} . This algorithm gives only a necessary condition to check $T(f) \neq false$. To obtain a sufficient condition we must add more constraints to the solution to force it to act like an upper probability. Using theorem 1 we must add all instances of L1,..,L4 where $k, m, n < n_{|\Omega|}$. For each of the instances of L4 there is an inequality of the form $r_1x_1 + \dots + r_Nx_N \geq k$. Let \hat{f} , the inequality formula corresponding to f , be the conjunction corresponding to \bar{f} together with all the inequalities corresponding to L1,..,L4 and the equations and inequalities $x_1 = 1$, $x_N = 0$ and $x_i \geq 0$ for $i \leq N$. From this we have:

Fact 4 $T(f) \neq false$ iff \hat{f} has a solution.

Theorem 4 If $(\forall X)(X \models f)$ then $T(f)$ is provable in **U-Sets**

Proof By corollary 1 $(\forall X)(X \models f)$ is equivalent to $T(f) = true$. I will prove the theorem in the form if $T(f) = false$ then $\neg T(f)$ is provable in **U-Sets**. If $T(f) = false$ then $(\forall X)(X \not\models f)$. Assume that $f = g_1 \vee \dots \vee g_n$ (where g_i is a conjunction of atomic formulae). Due to the fact that f is unsatisfiable every g_i is unsatisfiable. Therefore we can assume that $f = \bigwedge h_i$ where h_i is an atomic formula or the negation of an atomic formula and that $T(f) = false$. By fact 4 it holds that \hat{f} has no solutions. Due to the fact that the formulae of INEQ are provable in **U-Sets** it follows that the formula $T(\neg f'') = \neg T(f'')$ that results by replacing each instance of x_i in \hat{f} by $P^*(\rho_i)$ is **U-Sets** provable. All the conjuncts of $T(f'')$ that

are instances of axioms L1-L4 are **U-Sets** provable. Due to the fact that in intuitionistic sequent calculus from $\vdash \neg\alpha \vee \neg\beta$ and $\vdash \alpha$ we can prove $\vdash \neg\beta$ it follows that $\neg T(f)$ is provable in **U-Sets**. \square

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