A T-transitive opening of a proximity

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Abstract

A fast method to compute a T-transitive opening of a proximity is given for any t-norm, spending $O(n^3)$ time. Even though there is not a unique T-transitive approximation of a fuzzy relation, it is proved that the computed T-transitive opening is maximal.

Keywords: Fuzzy relation; transitive closure; T-indistinguishability; T- transitive opening..

1 Introduction

The classical concept of transitivity is generalised in fuzzy logic by the T-transitivity property of a fuzzy relations, where T is a continuous triangular t-norm.

Fuzzy relations on a finite set can also represent labelled directed graphs. Symmetric fuzzy relations can represent weighted complete undirected graphs where the set of nodes is the universe and the weighs of the edges are the relationship degrees. All triangle paths of a T-transitive graph are T-transitive.

A method to compute a low T-transitive approximation of fuzzy relations [Garmendia & Salvador; 2000] can be used to give new measure of T-transitivity of fuzzy relations. It can also be used to build T-transitive fuzzy relations from a given fuzzy relation. When the initial fuzzy relation is reflexive, the algorithm generates T-preorders that are different to the T-preorders generated form the T-transitive closure. The transitive closure of a fuzzy relation contains the initial relation, but the low T-transitive approximation relation is contained on the initial fuzzy relation.

There are T-transitive opening of a fuzzy relation, but in general the highest T-transitive opening cannot be found. This paper puts forward the existence of a maximal T-transitive opening from a reflexive and symmetric fuzzy relations, which is not unique, but there is not a T-transitive fuzzy relation that contains the opening and is contained in the fuzzy relation. An algorithm to compute it is given and it is proved that such transitive opening is maximal.

2 Preliminaries

Let $E = \{e_1, ..., e_n\}$ be a finite set.

Given a fuzzy relation $R: E\times E \to [0, 1]$ it is called e_{ij} to the relation degree value for elements e_i and e_j in E. So $e_{ij} = R(e_i, e_j)$.

A fuzzy relation R is **reflexive** if $e_{ii} = 1$ for $1 \le i \le n$.

The relation R is **symmetric** if $e_{ij} = e_{ji}$ for $1 \le i, j \le n$.

A binary operation $T: [0, 1] \times [0, 1] \to [0, 1]$ is a **t-norm** [Schweizer & Sklar; 1983] if it satisfies the following axioms:

1. $T(1, x) = x$
2. $T(x, y) = T(y, x)$
3. $T(x, T(y, z)) = T(T(x, y), z)$
4. If $x \le x'$ and $y \le y'$ then $T(x, y) \le T(x', y')$.

Definition 2.1. Let T be a triangular t-norm. A fuzzy relation $R: E\times E \to [0, 1]$ is **T-transitive** if

$T(R(a, b), R(b, c)) \leq R(a, c)$ for all a, b, c in E. So, $T(e_{ik}, e_{kj}) \leq e_{ij}$ for all i, j, k from 1 to n.

Definition 2.2. A reflexive and symmetric fuzzy relation is called a **proximity** relation. A **similarity** is a reflexive, symmetric and min-transitive fuzzy relation. A **T- indistinguishability** is a reflexive, symmetric and T-transitive fuzzy relation.

Definition 2.3. The relation A **includes** the relation B, (and it is denoted $A \supseteq B$) if $a_{ij} \geq b_{ij}$ for all i, j.

Definition 2.4. Bandler and Kohout [1988]

Let P be a property of fuzzy relations on an universe E. A fuzzy relation R_P is called the **P-opening** of a fuzzy relation R if:

1) R_P has property P
2) $R_P \subseteq R$
3) If R' has property P and $R_P \subseteq R' \subseteq R$ then $R_P = R'$.

Note that, according to this definition, a fuzzy relation R can have an infinite number of P-openings, even on a finite universe.

Definition 2.5. Given a reflexive and symmetric fuzzy relation R on a finite universe, the a **T-transitive opening** of R is a fuzzy T-indistinguishability relation R_T satisfying:

1) R_T is included in R ($R_T \subseteq R$)
2) If any fuzzy similarity relation H includes R_T and is included in R then it is R_T. (If $\exists$ H; $R_T \subseteq H \subseteq R$ then $H = R_T$).

Note that it can be several maximal T-transitive openings of a fuzzy relations.

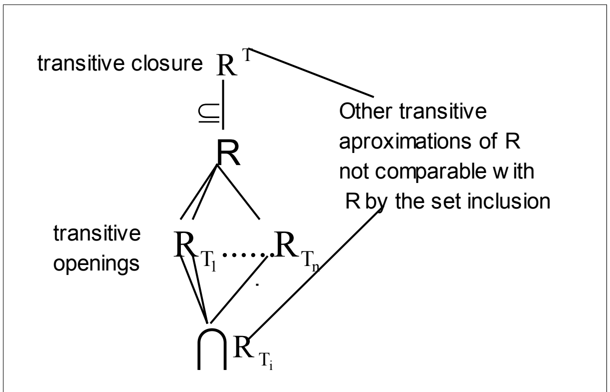

Figure 1: Relation of the T-transitive closure, T-transitive openings, and other T-transitive approximations not comparable by $\subseteq$.

Definition 2.6. The **residual implicator** J^T of a t-norm T, or quasi-inverse of T, is the binary operation J^T: $[0, 1]^2 \rightarrow [0, 1]$, defined by $J^T(x, y) = \mathrm{Sup}\{z \in [0, 1]; T(x, z) \leq y\}$.

In particular:

$$J^{Min}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y \end{cases}$$

$$J^{\mathrm{Prod}}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{if } x > y \end{cases}$$

$$J^{W}(x, y) = \min\{1 - x + y, 1\}$$

For a left-continuous t-norm T, and any x, y $\in$ [0, 1], it is known that $T(x, J^T(x, y)) \leq y$, i.e. $J^T(x, y)$ is the greatest solution, z, of the inequality $T(x, z) \leq y$. In case of a continuous t-norm T and $y \leq x$, then $J^T(x, y)$ is the greatest solution, z, of the equation $T(x, z) = y$.

3 Algorithm to compute a low T-transitive approximation of a proximity fuzzy relation

Let A be a proximity fuzzy relation.

Step 1. Set B initially blank

Step 2. Let U(A) be the set of elements of the upper triangular matrix of A sorted in a decreasing order.

Step 3. Set $b_{ii} = 1$, for all $1 \leq i \leq n$.

Step 4. While there is a blank in B do

Let a_{rs} be the highest value of the list U(A).

If b_{rs} is blank,

$I = \{j; b_{rj}$ is not blank in B$\}$ and $I' = \{i; b_{is}$ is not blank in B$\}$.

$H(A) = \{a_{ij}, i \in I, j \in I'$ sorted in increasing order$\}$.

While H(A) is not empty

Let a_{ij} be the smallest element not computed in H(A)

Set $b_{ij} = b_{ji} = \min\{a_{ij}, \min_k\{J^T(b_{ik}, b_{kj}), J^T(b_{jk}, b_{ki})\}$.

Delete the top element from H(A).

Delete the top element from U(A).

Example 1: Given the following approximation fuzzy relation:

$$A=\begin{pmatrix} 1 & 0.7 & 0.8 & 0.9 \\ 0.7 & 1 & 0.2 & 0.3 \\ 0.8 & 0.2 & 1 & 0.7 \\ 0.9 & 0.3 & 0.7 & 1 \end{pmatrix}$$

The algorithm is applied as follows to compute its low Min-transitive approximation B.

Step 1: Set B to be blank

Step 2: Let U(A) be the set of elements of the upper triangular matrix of A sorted in a decreasing order. U(A) = {0.9; 0.8; 0.7; 0.7; 0.3; 0.2}.

Step 3: Set b_{ii} =1 for all i.

Step 4: The greatest value of U(A), $a_{14} = 0.9$ is taken. Let I = {j; b_{1j} that are not blank values in matrix B} = {1} and let I' = {i; b_{i4} that are not blank in matrix B} = {4). The values $b_{41} = b_{14} = a_{14} = 0.9$ are inserted in B.

$$B=\begin{pmatrix} 1 & & & 0.9 \\ & 1 & & \\ & & 1 & \\ 0.9 & & & 1 \end{pmatrix}$$

The following greatest element in U(A) is 0.8 = a_{13}. I = {j; b_{1j} are not blank in B} = {1, 4} and I' = {i; b_{i3} is not blank in B } = {3). The values b_{13}, b_{43} and its symmetric values are inserted in B. H(A) = {a_{43}, a_{13}}, then $b_{43} = b_{34} = a_{43} = 0.7$, and $b_{13} = b_{31} = \min\{a_{13}, \min_k\{J^T(b_{1k}, b_{k3}\}, J^T(b_{3k}, b_{k1}\}\} = \min\{a_{13}, \min_k\{J^T(b_{14}, b_{43}\}, J^T(b_{34}, b_{41}\}\} = \min\{0.8, \{J^T(0.9, 0.7\}, J^T(0.7, 0.9\}\} = 0.7$.

$$B=\begin{pmatrix} 1 & & 0.7 & 0.9 \\ & 1 & & \\ 0.7 & & 1 & 0.7 \\ 0.9 & & 0.7 & 1 \end{pmatrix}$$

The following non blank greatest element in U(A) is 0.7 = a_{12}. I = {j; b_{1j} are not blank in B} = {1, 3, 4} and I' = {i; b_{i2} is not blank in B} = {2). H(A) = {a_{32}, a_{42}, a_{12}}. The values b_{12}, b_{32}, b_{42} and its symmetric values are inserted in B, having $b_{32} = a_{32} = 0.2$, $b_{42} = b_{24} = \min\{a_{42}, \min_k\{J^T(b_{4k}, b_{k2}\}, J^T(b_{2k}, b_{k4}\}\} = \min\{a_{42}, \min\{J^T(b_{43}, b_{32}\}, J^T(b_{23}, b_{34}\}\} = \min\{0.3, \min\{J^T(0.7, 0.2\}, J^T(0.2, 0.7\}\} = 0.2$, $b_{12} = b_{21} = \min\{a_{12}, \min\{J^T(b_{13}, b_{32}\}, J^T(b_{23}, b_{31}\}, J^T(b_{14}, b_{42}\}, J^T(b_{24}, b_{41}\}\} = 0.2$.

$$B_{Min}=\begin{pmatrix} 1 & 0.2 & 0.7 & 0.9 \\ 0.2 & 1 & 0.2 & 0.2 \\ 0.7 & 0.2 & 1 & 0.7 \\ 0.9 & 0.2 & 0.7 & 1 \end{pmatrix}$$

A Prod-transitive and W-transitive approximation given by this method are:

$$B_{Prod}=\begin{pmatrix} 1 & 0.257 & 0.777 & 0.9 \\ 0.257 & 1 & 0.2 & 0.285 \\ 0.777 & 0.2 & 1 & 0.7 \\ 0.9 & 0.285 & 0.7 & 1 \end{pmatrix}, \quad B_W=\begin{pmatrix} 1 & 0.4 & 0.6 & 0.9 \\ 0.4 & 1 & 0.2 & 0.3 \\ 0.6 & 0.2 & 1 & 0.7 \\ 0.9 & 0.3 & 0.7 & 1 \end{pmatrix}$$

Example 2: Given the following approximation fuzzy relation:

$$A=\begin{pmatrix} 1 & 0.1 & 0.2 & 0.5 \\ 0.1 & 1 & 0.4 & 0.1 \\ 0.2 & 0.4 & 1 & 0.1 \\ 0.5 & 0.1 & 0.1 & 1 \end{pmatrix}$$

The algorithm is applied to compute its low W-transitive approximation B, where W is the Łukasiewicz t-norm.

Step 1: Set B to be blank

Step 2: Let U(A) be the set of elements of the upper triangular matrix of A sorted in a decreasing order. U(A) = {0.5; 0.4; 0.2; 0.1; 0.1; 0.1}.

Step 3: Set b_{ii} =1 for all i.

Step 4: We take the greatest value of U(A), $a_{14} = 0.5$. Let I = {j; b_{1j} that are not blank values in matrix B} = {1} and let I' = {i; b_{i4} that are not blank in matrix B} = {4). The values $b_{41} = b_{14} = a_{14} = 0.5$ are inserted in B.

$$B=\begin{pmatrix} 1 & & & 0.5 \\ & 1 & & \\ & & 1 & \\ 0.5 & & & 1 \end{pmatrix}$$

The following greatest element in U(A) is 0.4 = a_{23}. I = {j; b_{2j} are not blank in B} = {2} and I' = {i; b_{i3} is not blank in B } = {3). The values b_{23} and b_{32} are inserted in B, having $b_{23} = b_{32} = 0.4$.

$$B = \begin{pmatrix} 1 & & & 0.5 \\ & 1 & 0.4 & \\ & 0.4 & 1 & \\ 0.5 & & & 1 \end{pmatrix}$$

The following non blank greatest element in U(A) is $0.2 = a_{13}$. I = {j; b_{1j} are not blank in B} = {1, 4} and I' = {i; b_{i3} is not blank in B } = {2, 3}. The values b_{12}, b_{13}, b_{42}, b_{43} and its symmetric values are inserted in B. The list $H(A) = \{a_{12}, a_{42}, a_{43}, a_{13}\}$, but $a_{12} = a_{42} = a_{43} = 0.1 < a_{13} = 0.2$. If the first element in H(A) is $a_{12} = 0.1 = b_{12}$, then $b_{42} = b_{24} = \min\{a_{42}, \min_k\{J^T(b_{4k}, b_{k2}\}, J^T(b_{2k}, b_{k4}\}\} = \min\{0.1, \min\{J^W(0.5, 0.1\}, J^W(0.1, 0.5\}\} = 0.1$, $b_{43} = b_{34} = \min\{a_{43}, \min\{J^T(b_{42}, b_{23}\}, J^T(b_{32}, b_{24}\}\} = \min\{0.1, \min\{J^W(0.1, 0.4\}, J^W(0.4, 0.1)\}\} = 0.1$, $b_{13} = b_{31} = \min\{a_{13}, \min\{J^T(b_{12}, b_{23}\}, J^T(b_{32}, b_{21}\}, J^T(b_{14}, b_{43}\}, J^T(b_{43}, b_{14}\}\} = \min\{0.2, \min\{1, 0.7, 0.6, 1\}\} = 0.2$, so:

$$B_W = \begin{pmatrix} 1 & 0.1 & 0.2 & 0.5 \\ 0.1 & 1 & 0.4 & 0.1 \\ 0.2 & 0.4 & 1 & 0.1 \\ 0.5 & 0.1 & 0.1 & 1 \end{pmatrix},$$

A Prod-transitive and Min-transitive approximation given by this method are:

$$B_{Prod} = \begin{pmatrix} 1 & 0.1 & 0.2 & 0.5 \\ 0.1 & 1 & 0.4 & 0.1 \\ 0.2 & 0.4 & 1 & 0.1 \\ 0.5 & 0.1 & 0.1 & 1 \end{pmatrix}, B_{Min} = \begin{pmatrix} 1 & 0.1 & 0.1 & 0.5 \\ 0.1 & 1 & 0.4 & 0.1 \\ 0.1 & 0.4 & 1 & 0.1 \\ 0.5 & 0.1 & 0.1 & 1 \end{pmatrix}.$$

Lemma 1. The output of the algorithm applied to a reflexive and symmetric fuzzy relation is a T-indistinguishability fuzzy relation.

Proof. The proof is trivial by the construction method.

The following lemma show that the previous algorithm gives a T-transitive opening of a reflexive and symmetric fuzzy relation.

Lemma 2. Let A be a reflexive and symmetric fuzzy relation, and let B be the output of the previous algorithm applied to A. If any T-indistinguishability fuzzy relation H includes B and is included in A then it is B (if $\exists$ H | $B \subseteq H \subseteq A$ then H = B).

Proof.

Suppose that there is a T-indistinguishability $H = (h_{ij})$ such that $B \subseteq H \subseteq A$. So $b_{ij} \le h_{ij} \le a_{ij}$ and $\max_k\{T(h_{ik}, h_{kj})\} \le h_{ij}$, for all i, j.

If $B \ne H$ then $\exists$ (r, s) such that $b_{rs} < h_{rs} \le a_{rs}$.

Let (r, s) be the indexes of the first computed b_{rs} by the algorithm such that $b_{rs} < h_{rs} \le a_{rs}$. Let I, I' be the set of indexes given by the algorithm in the step 4 in which b_{rs} is generated. Then $(r, s) \in I \times I'$. Then it is known that $b_{rs} = \min\{a_{rs}, \min_j\{J^T(b_{rj}, b_{js}), J^T(b_{sj}, b_{jr})\}\}$ where $r \in I$ and $s \in I'$ and b_{ij} are already computed in B and $\max_k\{T(h_{rk}, h_{ks})\} \le h_{rs} \le a_{rs}$. $\text{Min}_j\{J^T(b_{rj}, b_{js}), J^T(b_{sj}, b_{jr})\}$ is a maximal solution, because $\sup\{z, T(x, z) \le y\} = J^T(x, y)$, and (r, s) is the first indexees with $b_{rs} < h_{rs} \le a_{rs}$, so the computed values in B are equal to the values of H, and so $\max_k\{T(h_{rk}, h_{ks})\} \le h_{rs} \le \max_j\{T(h_{rj}, h_{js})\} \le \max_j\{T(b_{rj}, b_{js})\} \le b_{rs} = \min\{a_{rs}, \min_j\{J^T(b_{rj}, b_{js}), J^T(b_{sj}, b_{jr})\}\}$ which is contradictory to (1). Thus, any fuzzy similarity H such that $B \subseteq H \subseteq A$ verifies that H = B.

Therefore, the algorithm computes a maximal T-indistinguishability from a reflexive and symmetric fuzzy relation, which is a T-transitive opening.

Lemma 3. The low T-transitive approximation of a reflexive and symmetric fuzzy relation can be computed in $O(n^3)$ time in the worst case.

Proof

The computational complexity of the time consumed by the algorithm is analysed as follows:

Step 2 sorts $\frac{n^2}{2} - n$ values, so takes $O(n^2 \log n)$ time.

Step 4: A general proof of the complexity time of the step four is done by computing the complexity in the worst case, which is the case in which it is needed to fill n-1 blocks of dimension one. Suppose that for each value in U(A) it must be added a dimension one block, then it must be constructed n-1 blocks in B, from k=2 to n, and each building must sort the (k-1) elements in H(A), and then make the k-1 assignments in B. Then the final complexity time of the step four in the order of

$$\sum_{k=2}^{n} (\text{sort of k-1 elements in H(A)}) + (\text{assignments of k-1 el}$$

It is, the step four the complexity time is computed as follows:

$$O(\sum_{k=2}^{n}\left(((\sum_{j=1}^{k-1} j) \right)) = O(\sum_{k=2}^{n}\left(\frac{(k-1)+1}{2}(k-1) \right)) =$$

$$O(\sum_{k=2}^{n}\left(k^2\right)) = O(n^3),$$

which is the complexity time of the algorithm.□

5 Conclusions

Its is given a O(n^3) time algorithm to compute a T-transitive opening of a proximity given any t-norm T.

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References

[1]. C. Alsina, E. Trillas, L. Valverde, L. On some logical connectives for fuzzy set theory, J. Math. Ann. Appl. 93 (1983) 15-26.

[2]. W. Bandler, J. Kohout, Special properties, closures and interiors of crisp and fuzzy relations. Fuzzy Sets and Systems 26 (1988) 317-331.

[3]. D. Boixader, On the relationship between T-transitivity and approximate equality Fuzzy Sets and Systems 33 (2003) 6 – 69.

[4]. B. De Baets, De Meyer, H., Transitive approximation of fuzzy relations by alternating closures and openings, Soft Computing 7 (2003) 210-219.

[5]. F. Esteva, P. Garcia, L. Godo, R.O. Rodriguez, Fuzzy approximation relations, modal structures and possibilistic logic, Mathware and Soft Computing 5 (2-3) (1998) 151-166.

[6]. L. Garmendia, C. Campo, S. Cubillo, A. Salvador, A Method to Make Some Fuzzy Relations T-Transitive. I. J. of Intelligence Systems. Vol. 14, Nº 9, (1999) 873 – 882.

[7]. L. Garmendia, A. Salvador, A. On a new method to T-transitivize fuzzy relations, Information Processing and Management of Uncertainty in Knowledge - based Systems, IPMU 2000. (2000) 864 – 869.

[8]. L. Garmendia, A. Salvador, On a new method to T-transitivize fuzzy relations, in Technologies for Constructing Intelligent Systems 2, Springer. Edited by Bouchon-Meunier, B., Gutierrez-Rios, J., Magdalena, L., Yager, R. R, (2000) 251 – 260.

[9]. J. Jacas, On the generators of T-indistinguishability operators. Schocastica 12 (1988) 49-63.

[10]. J. Jacas, Similarity relations. The calculation of minimal generating families. Fuzzy Sets and Systems 35 (1990) 151-162.

[11]. J. Jacas, J. Recasens, Fuzzy T-transitive relations: eigenvectors and generators, Fuzzy Sets and Systems 72 (1995) 147–154.

[12]. J. Jacas, J. Recasens, Decomposable indistinguishability operators, Proceedings of the Sixth IFSA Congress, Sao Paulo, 1995.

[13]. H. S. Lee, An optimal algorithm for computing the max–min transitive closure of a fuzzy similarity matrix , Fuzzy Sets and Systems 123 (2001) 129–136.

[14]. R.O. Rodriguez, F. Esteva, P. Garcia, L. Godo, On implicative closure operators in approximate reasoning. International Journal of Approximate Reasoning 33 (2003) 159-184.

[15]. B. Schweizer, A. Sklar A. Probabilistic Metric Spaces, North-Holland, New York, (1983).