Multiresolution Analysis and Fuzzy Indistinguishability Operators: a first approach

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Abstract

There exists a strong conceptual relation between Multiresolution Analysis, a theory mainly used in Wavelet Theory, and Indistinguishability Operators, a theory mainly used in Fuzzy Sets Theory. In this work a first approach to find a formal link between both theories is tried. It will be shown that, in some cases, a multiresolution analysis allows to build a family of Indistinguishability Operators with an analogous axiomatic framework.

Keywords: Multiresolution Analysis, Indistinguishability Operators, Wavelet Theory, Fuzzy Set Theory.

1 Introduction

The theory of Multiresolution Analysis, which was introduced by Mallat in [6], is usually used in Wavelet Theory as the main method to build wavelet functions. It consists of a family of nested functional subspaces which is dense in the functional space and has empty intersection. The projections of a function in each subspace give an approximation to this function with arbitrary accuracy. But an approximation to a function in a functional subspace implies a capacity of distinguishing: the approximation considers equal all functions with differences smaller than its accuracy level. In this sense, each level in a multiresolution analysis implicitly has an associated indistinguishability operator on the elements of the functional space.

A multiresolution analysis requires some other, and perhaps more important, properties. In first place, it must be possible to go from one level to another by means of dyadic expansions or contractions. In second place, there must exist a function, named scaling function, which generates by integer shifts and dilations a basis of the functional subspaces. There are several such functions, especially B-spline functions, which can be fuzzy sets and generate multiresolution analysis. These will be the most interesting functions in our study.

Our interest focus on Fuzzy Sets Theory and, in particular, in the field of Indistinguishability Operators [9, 5] so we want to obtain families of indistinguishability operators on the universe of discourse through an adequate Multiresolution Analysis. In a previous work [3], a strong relation between a family of Indistinguishability Operators and a hierarchical family of fuzzy partitions was shown. The family of these fuzzy partitions are a part of a known multiresolution analysis.

The paper is structured as follows. First a light introduction to both theories, Multiresolution Analysis and Indistinguishability Operators is given. Only the properties which are used later will be considered. Third section is dedicated to obtaining several families of Indistinguishability Operators using a scaling function. Finally, an axiomatization of a multiresolution family of Indistinguishability Operators is proposed as a research line for future works.

2 Multiresolution Analysis

A Multiresolution Analysis is based on a sequence of successive approximation functional spaces such as

$$\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots \quad (1)$$

with

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}) \quad (2)$$

and

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}. \quad (3)$$

where $L^2(\mathbb{R})$ is the space of all square integrable functions on the real space $\mathbb{R}$.

The spaces V_i are used to approximate general functions and since the union of all of them is dense in $L^2(\mathbb{R})$, any given function can be approximated arbitrarily close by means of its projections $P_j f$ on V_j. It is clear that each subspace V_j bears a resemblance with the idea of an indistinguishability measure, because if the functions $f, g \in L^2(\mathbb{R})$ have equal projections $P_j f = P_j g$ over the subspace V_j they are indistinguishable with respect to V_j.

Multiresolution analysis is mainly used in Wavelet Theory [2] to build wavelet families and, for that purpose, three additional requirements are needed:

$$f \in V_j \Leftrightarrow f(2^j \cdot) \in V_0. \quad (4)$$

that is, all the spaces are scaled versions of the central space V_0,

$$f(\cdot) \in V_0 \Rightarrow f(\cdot - n) \in V_0 \qquad \forall n \in \mathbb{Z}, \quad (5)$$

V_0 is closed under integer translations and

$$\exists \phi \in V_0 \ : \ \{\phi_{0,n} \ : \ n \in \mathbb{Z}\} \text{ is a Riesz basis of } V_0 \quad (6)$$

i.e. there exists a function ϕ, named scaling function, which generates by integer translations an orthonormal base of V_0 and together with (4) it results that $\{\phi_{jn} = \phi(2^j \cdot - n) \ : \ n \in \mathbb{Z}\}$ is a orthonormal basis of V_j. Last requirement is fundamental to multiresolution analysis and since $\phi \in V_0 \subset V_1$ a square summable sequence c_k exists such that the scaling function satisfies:

$$\phi(x) = \sum_k c_k \phi(2x - k). \quad (7)$$

Last equation is named the refinement equation and the function ϕ is said to be a refinable function.

As can be seen in [2], the scaling function can be used as a starting point to build a multiresolution analysis. In this case, a function ϕ satisfying (7) is chosen and subspaces V_j are defined as the closed subspace spanned by $\phi_{jk}, k \in \mathbb{Z}$ with $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k)$. Only two conditions on c_k are necessary and sufficient to ensure that $\{\phi_{jk}, k \in \mathbb{Z}\}$ is a Riesz basis in each V_j and that the subspaces V_j are nested:

1. $\sum |c_k|^2 < +\infty$
2. $0 < \alpha \leq \sum_{l\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi l)|^2 \leq \beta < +\infty$

where $\hat{\phi}$ is the Fourier transform of ϕ and ξ is the variable on frequency domain. Moreover, if ϕ is continuous and satisfies $|\phi(x)| \leq C(1 + |x|)^{-1-\epsilon}$ and $\sum_{l\in\mathbb{Z}} \phi(x - l) = k \neq 0$, which implies both $\phi \in L^1$ and $\int \phi(x)dx \neq 0$, the conditions (2) and (3)are satisfied and then a full multiresolution analysis is obtained. These conditions are equivalent to $\sum c_{2n} = \sum c_{2n+1} = 1$.

2.1 B-Splines

An interesting example of a multiresolution analysis is given by the space S_r of cardinal spline functions of order r in $L^2(\mathbb{R})$. A cardinal spline is a piecewise polynomial function defined on $\mathbb{R}$, of local degree less than r, that has breakpoints at the integers and global smoothness in C^{r-2}. Let $V(\phi)$ be the shift-invariant subspace of $L^2(\mathbb{R})$ generated by ϕ, i.e the closure of the space of all finite linear combinations of ϕ and its integer shifts $\phi(\cdot + j)$. Then, the space S_r of cardinal spline functions of order r is generated by cardinal B-spline that has knots at $0, 1, \ldots, r$. These B-splines are easy to define recursively:

$$\begin{aligned}\beta_1 &= \mathcal{X}_{[0,1)}, \text{ the Haar function}\\ \beta_r &= \beta_{r-1} * \beta_1\end{aligned}$$

where $*$ is the convolution operator defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt.$$

A explicit formula is given by:

$$\beta_n(x) = \sum_{j=0}^{n+1} \frac{(-1)^j}{n} \binom{n+1}{j} (x-j)^n \mu(x-j),$$

where μ is the function such that $\mu(x) = 1$ when $x \geq 0$ and $\mu(x) = 0$ otherwise. For example, β_2 is the classical triangular fuzzy set (or hat function), β_3 is a C^1 piecewise quadratic function and so on.

The refinement equation for B-splines of order r is given by:

$$\beta_r(x) = 2^{-r+1} \sum_{j=0}^{r} \binom{r}{j} \beta_r(2x-j).$$

It is possible to define symmetric B-spline functions centered on the origin by introducing the appropriate offset: $(n+1)/2$. Their expression is:

$$\mathring{\beta}_n(x) = \sum_{j=0}^{n+1} \frac{(-1)^j}{n} \binom{n+1}{j} \cdot (x + \frac{n+1}{2} - j)^n \mu(x + \frac{n+1}{2} - j).$$

and the coefficients of the refinement equation when n is odd are:

$$c_k(n) = \begin{cases} \frac{1}{2^n}\binom{n+1}{k+(n+1)/2} & |k| \leq (n+1)/2 \\ 0 & \text{otherwise} \end{cases}.$$

3 Indistinguishability Operators

A T-indistinguishability operator $E : X \times X \longrightarrow [0,1]$ is a fuzzy relation which is reflexive, symmetric and T-transitive, being T a t-norm [7]. Given a continuous t-norm T, the natural T-indistinguishability E_T on $[0,1]$ is defined by

$$E_T(x,y) = T(\hat{T}(x,y), \hat{T}(y,x)),$$

where $\hat{T}$ is the residuation operator

$$\hat{T}(x,y) = \sup\{z \in [0,1] \ : \ T(x,z) \leq y\}.$$

T-indistinguishability operators have been widely studied in the setting of Fuzzy Set Theory, see for example [9, 5].

The natural T-indistinguishability operators on $[0,1]$ associated with the continuous t-norms minimum (min), product ($\prod(x,y) = xy$) and Lukasiewicz ($W(x,y) = max(0, x+y-1)$) are:

$$\begin{aligned} E_{\min}(x,y) &= \begin{cases} 1 & \text{if } x = y \\ \min(x,y) & \text{otherwise} \end{cases}, \\ E_{\prod}(x,y) &= \min(\frac{x}{y}, \frac{y}{x}), \\ E_W(x,y) &= 1 - |x-y|. \end{aligned}$$

Given a fuzzy subset μ in a set X, the fuzzy relation defined by

$$E_\mu(x,y) = \hat{T}(\max(\mu(x).\mu(y)), \min(\mu(x), \mu(y)))$$

is a T-indistinguishability operator on X.

4 Binary relations through scaling functions

Given a scaling function ϕ with range in $[0,1]$ it is possible to define a binary relation on $\mathbb{R}$ by means of $E(x,y) = \phi(x-y)$. To be a T-indistinguishability operator, E should satisfy the following conditions:

1. reflexivity: $E(x,x) = \phi(x-x) = \phi(0) = 1$,
2. symmetry: $E(x,y) = E(y,x) \Leftrightarrow \phi$ is symmetrical function with respect to 0.
3. T-transitivity with respect to some t-norm T

Postponing the transitivity property, it would be possible to define a family of binary relations as follows:

$$E^i(x,y) = \phi^i(x-y) = \phi(2^i(x-y)) = E^{i-1}(2x, 2y).$$

The refinement equation gives another relation between E^0 and E^1:

$$\begin{aligned} E^0(x,y) &= \phi(x-y) \\ &= \sum_k c_k \phi(2(x-y)-k) \\ &= \sum_k c_k \phi(2(x-\frac{k}{2}-y)) \\ &= \sum_k c_k \phi(2(x-(y+\frac{k}{2})) \\ &= \sum_k c_k E^1(x-\frac{k}{2},y) \\ &= \sum_k c_k E^1(x,y+\frac{k}{2}). \end{aligned}$$

As an important example, $\mathring{\beta}_1(x) = max(0, 1-|x|)$ is a fuzzy triangular number and symmetrical B-spline of order 1. The relation $E_1(x,y) = \phi(x-y) = max(0, 1-|x-y|) = E_W(x,y)$ is a W-indistinguishability operator. Since the refinement equation for $\mathring{\beta}_1^c$ is:

$$\begin{aligned} \mathring{\beta}_1^0 &= \frac{1}{2}\mathring{\beta}_1^1(x-\frac{1}{2}) + \mathring{\beta}_1^1(x) + \frac{1}{2}\mathring{\beta}_1^1(x+\frac{1}{2}) \\ &= \frac{1}{2}\mathring{\beta}_1^0(2x-1) + \mathring{\beta}_1^0(2x) + \frac{1}{2}\mathring{\beta}_1^0(2x+1), \end{aligned}$$

the relations $E_1^i(x,y) = max(0, 1-2^i|x-y|)$ verify:

$$\begin{aligned} &E_1^0(x,y) = \mathring{\beta}_1^0(x-y) \\ &= \frac{1}{2}E_1^1(x-\frac{1}{2},y) + E_1^1(x,y) + \frac{1}{2}E_1^1(x,y-\frac{1}{2}) \\ &= \frac{1}{2}E_1^0(2x-1,2y) + E_1^0(2x,2y) \\ &\quad +\frac{1}{2}E_1^0(2x,2y-1) \end{aligned}$$

We can generalize this result.

Theorem Let E be a T-indistinguishability operator defined on a real space X closed under dilations, i.e., if $x \in X$ also $2x \in X$, then the relations $E^i(x,y) = E(2^i x, 2^i y)$ are T-indistinguishability operators.

Proof Reflexivity and symmetry properties are evident.T-transitivity is easily verified

$$\begin{aligned} &T(E^i(x,y), E^i(y,z)) \\ &\quad = T(E(2^i x, 2^i y), E(2^i y, 2^i z)) \\ &\quad \leq E(2^i x, 2^i z) \\ &\quad = E^i(x,z). \end{aligned}$$

So all relations E_1^i are W-indistinguishability operators.

The last case is not valid for all centered B-splines. Let $\mathring{\beta}_0$ be the Haar function or B-spline of order 0 define by

$$\mathring{\beta}_0(x) = \begin{cases} 1 & x \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & otherwise \end{cases}$$

and let $\mathring{\beta}_0^i$ be the functions defined by $\mathring{\beta}_0^i(x) = \mathring{\beta}_0(2^i x)$. Let us consider the binary relations $E_0^i(x,y) = \mathring{\beta}_0^{-i}(x-y)$. They are reflexive and symmetric relations but they are not T-transitive for any t-norm T. For any E_0^i with $i \in \mathbb{Z}$ there exists three values $x_0 = 0, y_0 = 2^{-i-1}, z_0 = 2^{-i} + \epsilon$ with $0 < \epsilon < 2^{-i-1}$ such that $|x-y| \leq 2^{-i}, |y-z| \leq 2^{-i}$ and $|x-z| > 2^{-i}$, and hence $E_0^i(x,y) = E_0^i(y,z) = 1$ and $E_0^1(x,y) = 0$. The Z t-norm defined by

$$Z(x,y) = \begin{cases} y & x = 1 \\ x & y = 1 \\ 0 & otherwise \end{cases}$$

is the smallest t-norm. If a binary relation is T-transitive with respect to a t-norm then it is Z-transitive, but it is clear that E_0^i are not Z-transitive relations because $Z(E_0^i(x_0,y_0), E_0^i(y_0,z_0)) > E_0^i(x_0,z_0)$.

This result is essentially the Poincare's Paradox relative to "physical continuum" (see [7] page 223) and transitive relations. Poincare saw as an unacceptable property that an object A could not be distinguished form another object B, B could not distinguished from another object C and yet A could be distinguished from C. But this happens in many situations; for instance all measure devices have a accuracy level and hence they suffer this paradox.

In spite of the families E_1^i and E_0^i are different because of the transitivity property, they have interesting common properties.

Proposition The families of reflexive and symmetric relations $\{E_1^i\}$ and $\{E_0^i\}$ verify the properties:

$$\lim_{\substack{i\to\infty \\ x\neq y}} E^i(x,y) = 0 \tag{8}$$

$$\lim_{i\to-\infty} E^i(x,y) = 1 \tag{9}$$

$$E^i \geq E^{i+1} \tag{10}$$

Proof Let us prove it for E_0^i. As $2^{-i} < 2^{-i+1}$ we have

$$-2^{-i} < -2^{-i-1} \leq x - y \leq 2^{-i-1} < 2^{-i}$$

and so $E_0^i \geq E_0^{i+1}$. Given $x, y \in \mathbb{R}$, there always exists an $n_0 \in \mathcal{N}$ such that $|x-y| > 2^{-n_0}$ and hence $E^i(x,y) = 0$ for all $i \geq n_0$. That proves (8). Similarly, for all $x, y \in \mathbb{R}$ there always exists an $n_0 \in \mathcal{N}$ such that $|x-y| < 2^{n_0}$ so $E^i(x,y) = 1$ for all $i \leq -n_0$, and thus (9) is proved.

For E_1^i, properties (8,9) are proved by continuity. Finally, as $2^i|x-y| \leq 2^{i+1}|x-y|$ then $1-2^i|x-y| \geq 1-2^{i+1}|x-y|$ and $E^i(x,y) = max(0, 1-2^i|x-y|) \geq max(0, 1-2^{i+1}|x-y|) = E^{i+1}(x,y)$.□

The families of binary relations E_0^i and E_1^0 have some analogous properties to multiresolution analysis, but all E_1^i are W-indistinguishability operators while E_0^i are only reflexive and symmetric binary relations and not T-transitive for any t-norm T. With respect to Poincare's Paradox, next theorem might be interesting.

Theorem For any $x, y, z \in \mathbb{R}$, there exists an $i \in \mathbb{Z}$ such that

$$\min(E_0^i(x,y), E_0^i(y,z)) \leq E_0^i(x,y).$$

Proof Let $i_0 \in \mathbb{N}$ be a natural number such that $2^{-i_0} < \min(|x-y|, |y-z|, |x-z|)$, then it is evident that $E_0^{i_0}(x,y) = E_0^{i_0}(y,z) = E_0^{i_0}(x,z) = 0$.□

A relation E defined by $E(x,y) = \phi(x-y)$ for some function ϕ will be a reflexive relation if $\phi(0) = 1$ and a symmetrical one if $\phi(x) = \phi(-x)$, but when will it be T-transitive for some t-norm T?

If we consider the min t-norm, we have the inequality

$$\min(\phi(x-y), \phi(y-z)) \leq \phi(x-z)$$

Trying to resolve this functional equation [1], we take $x = z$ and then

$$\begin{aligned} &\min(\phi(x-y), \phi(y-x)) \\ &\quad = \min(\phi(x-y), \phi(x-y)) \\ &\quad = \phi(x-y) \\ &\quad \leq \phi(0) = 1, \end{aligned}$$

as the domain of $x-y$ is $\mathbb{R}$, we have $\phi(t) \leq 1$ for all $t \in \mathbb{R}$. Taking $x = 0$ we obtain

$$\begin{aligned} \min(\phi(-y), \phi(y-z)) &= \min(\phi(y), \phi(y-z)) \\ &\leq \phi(-z) \\ &= \phi(z), \end{aligned}$$

and so

$$\min(\phi(y), \phi(z-y)) \leq \phi(z).$$

Let z be such that $z - y = y$, then

$$\min(\phi(y), \phi(y)) = \phi(y) \leq \phi(2y),$$

and for all $y \in \mathbb{R}$, $\phi(y) \leq \phi(2y)$. Adding the continuity on ϕ we get that $\phi(x) = 1$ for all $x \in \mathbb{R}$. It is enough to take $y = \frac{x}{2^n}$ and then $\phi(\frac{x}{2^n}) \leq \phi(\frac{x}{2^{n-1}}) \leq \cdots \leq \phi(x) \leq 1$ and by continuity $\lim_{n\to\infty} \phi(\frac{x}{2^n}) = \phi(0) = 1$.

When we take an archimedian t-norm, continuity is not a sufficient condition to determinate a full expression for ϕ. A similar problem was studied in [4] but with an inverse approach. There, families of fuzzy numbers, invariant under translations and singletons of a T-indistinguishability operator were studied. Under these conditions and with T an archimedian t-norm, these operators take the expression

$$E(x,y) = t^{[-1]} \circ F(y-x),$$

where t is the additive generator of T, $t^{[-1]}$ is its pseudo-inverse and F a sub-additive function with $F(0) = 0$, even and non decreasing in $(0, +\infty)$.

In our case, the relations E^i will have all their columns with the same shape because $E_x^i(y) =$

$\phi^i(x-y) = \phi(2^i(x-y))$. If we consider the family of all columns of each E^i, this family is invariant under translations in the sense defined in[4], because of $E_a(x) = E_b(x-a+b)$ so we have a similar case.

5 Axiomatization of a multiresolution family of Indistinguishability Operators

We finalize this work with a proposal of axiomatization of the concept of a multiresolution family of T-indistinguishability Operators.

Definition A family of binary relations $E^i : i \in \mathbb{Z}$ is a multiresolution family if it verifies:

1. $\lim_{\substack{i\to\infty \\ x\neq y}} E^i(x,y) = 0$
2. $\lim_{i\to-\infty} E^i(x,y) = 1$
3. $E^i \geq E^{i+1}$
4. $E^i(x,y) = E^{i-1}(2x,2y)$.

If all E^i are T-indistinguishability operators for some t-norm T, then we have a multiresolution family of T-indistinguishability operators.

The definition tries to capture the main properties of a multiresolution system using T-indistinguishability operators. In a have a multiresolution system we have different levels in our capacity of distinguishing. The system is capable of distinguishing any couple of values and no couple of values is distinguishable in every level. Moreover we can calculate the indistinguishability value of a couple of values in terms of indistinguishability values in another level.

In this work we have shown an example of a multiresolution family of W-indistinguishability operators E_1^i and a multiresolution family of reflexive and symmetrical relations but which are not T-transitive for any t-norm, E_0^i. In future works we would like to characterize this type of binary relation families.

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