THE VAGUE SUM AND PRODUCT OF A FINITE NUMBER OF REAL NUMBERS IN VAGUE ARITHMETIC

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Abstract

The vague sum (product) of a finite number of real numbers and the vague integral multiples (powers) of real numbers based on a perfect vague addition (multiplication) operation are introduced, and some of their fundamental properties are pointed out. Furthermore, some examples are designed to show how they can be practically implemented.

Keyword: Fuzzy arithmetic, Vague arithmetic, Vague arithmetic operation, Many-valued equivalence relation, Fuzzy equivalence relation, Indistinguishability operator.

1 INTRODUCTION

Vague arithmetic has been introduced in [4, 5] to model the vaguely defined arithmetic operations on the set $\mathbb{R}$ of all real numbers. Many-valued equivalence relations (indistinguishability operators, fuzzy equivalence relations, similarity relations, etc.) [2, 3, 9, 10, 11, 12] on $\mathbb{R}$ are natural mathematical tools for the representation of the uncertainty resulting from the indistinguishability of any two real numbers in practical situations, especially in a practical measurement process [6]. In contrast to usual arithmetic, vague arithmetic basically takes into account this kind of uncertainty and its mathematical representation, i.e. many-valued equivalence relations, and establishes the vaguely defined arithmetic operations (the so-called vague arithmetic operations, i.e. the vague addition, multiplication, subtraction and division operations [4, 5]) by means of many-valued equivalence relations. Whenever a vague addition $\tilde{+}$ (multiplication $\tilde{\cdot}$) operation on $\mathbb{R}$ is taken into consideration, the formulation of the notions of the sum (product) of a finite number of real numbers and the integral multiples (powers) of real numbers in terms of $\tilde{+}$ ($\tilde{\cdot}$), which will be called the vague sum (product) of a finite number of real numbers and the vague integral multiples (powers) of real numbers respectively, possess a significant concern for the practical implementation of vague arithmetic. An abstract approach to the solution to this problem within the framework of vague groups has been proposed in [7, 8]. Since this problem is handled in an abstract mathematical structure-vague group, its solution requires some additional mathematical tools and sophisticated techniques. The main aim of this paper is to simplify this solution by restricting $\tilde{+}$ ($\tilde{\cdot}$) to a perfect vague addition (multiplication) operation on $\mathbb{R}$, and to establish the vague sum (product) of a finite number of real numbers and the vague integral multiples (powers) of real numbers without needing vague algebraic notions. The other aim of this paper is to show how the present notions can be practically applied.

2 THE PERFECT VAGUE ADDITION AND MULTIPLICATION OPERATIONS

The aim of this preliminary section is to give the definitions of the perfect vague addition and multiplication operations, and to supply the necessary background materials. We first need to introduce the many-valued logical base of this study. A 5-tuple $M = (L, \leq, *, \mathbf{0}, \mathbf{1})$ is called a partially ordered monoid (for short, po-monoid) with the zero element $\mathbf{0}$ [1] provided with the following axioms:

(i) $(L, \leq, *)$ is a partially ordered groupoid, i.e. $(L, \leq)$ is a partially ordered set, and $*$ is an isotonic binary operation on L, i.e. for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$,

$$((\alpha_1 \leq \alpha_2) \textit{ and } (\beta_1 \leq \beta_2)) \Rightarrow (\alpha_1 * \beta_1 \leq \alpha_2 * \beta_2)$$

(ii) $\mathbf{0}$ and $\mathbf{1}$ are, respectively, the least element and the greatest element of $(L, \leq)$,

(iii) $(L, *, \mathbf{1})$ is a commutative monoid.

The elements of L can be interpreted as truth values. For $L = [0,1]$, the monoid operation $*$ is known as a t-norm in fuzzy logic. Throughout this paper, the 5-tuple $M = (L, \leq, *, \mathbf{0}, \mathbf{1})$ always stands for a fixed po-monoid with the zero element $\mathbf{0}$ different from $\mathbf{1}$, and forms the many-valued logical base of this study. A map $\mu : X \to L$ is called an L-fuzzy set of X, and the set of all L-fuzzy sets of X is denoted by L^X. An L-fuzzy set ρ of $X \times Y$ is called a fuzzy relation from X to Y. Given a fuzzy relation $\tilde{\circ}$ from $X \times X$ to X, for $x,y,z \in X$, $\tilde{\circ}((x,y),z)$ will be denoted by $\tilde{\circ}(x,y,z)$ for the sake of simplicity. An M-equivalence relation on X [3] is a map $E : X \times X \to L$ satisfying the following three axioms:

(E.1) $E(x,x) = \mathbf{1}$, $\forall x \in X$,

(E.2) $E(x,y) = E(y,x)$, $\forall x,y \in X$,

(E.3) $E(x,y) * E(y,z) \leq E(x,z)$, $\forall x,y,z \in X$.

An M-equivalence relation E on X is called an M-equality on X iff E additionally fulfills the separation axiom:

(E.1$'$) $E(x,y) = \mathbf{1} \Rightarrow x = y, \quad \forall x,y \in X$.

An M-equivalence relation E on $\mathbb{R}$ is said to be regular w.r.t. the usual addition (multiplication) operation "$+$" ("$.$") [5] iff for all $x,u,y \in \mathbb{R}$,

$$E(x,y) \leq E(x+u, y+u) \; (E(x,y) \leq E(x.u, y.u)).$$

Perfect fuzzy functions are the essential notions to define the perfect vague addition and multiplication operations. For this reason, it will be helpful to recall these notions at first:

Definition 1 *[3]Let E and F be two M-equivalence relations on X and Y, respectively. A fuzzy relation $\rho \in L^{X \times Y}$ is called a perfect fuzzy function from X to Y w.r.t. E and F iff ρ satisfies the following conditions:*

(i) $\rho(x,y) * E(x,x') \leq \rho(x',y)$, $\forall x,x' \in X$, $\forall y \in Y$,

(ii) $\rho(x,y) * F(y,y') \leq \rho(x,y')$, $\forall x \in X$, $\forall y,y' \in Y$,

(iii) $\rho(x,y) * \rho(x,y') \leq F(y,y')$, $\forall x \in X$, $\forall y,y' \in Y$,

(iv) For all $x \in X$, $\exists y \in Y$ such that $\rho(x,y) = \mathbf{1}$.

The perfect fuzzy function is a slightly stronger version of the notion of fuzzy function introduced in [2, 10], and possesses powerful representation properties by means of ordinary functions and many-valued equivalence relations. Before introducing these properties, we need the concept of extensionality of an ordinary function $f : X \to Y$ according to given M-equivalence relations E on X and F on Y, which can be simply stated as the satisfaction of the inequality

$$E(x,x') \leq F(f(x), f(x')), \; \forall x,x' \in X.$$

Extensionality of an ordinary function can be informally expressed as the satisfaction of the intuitive expectation "If two points are similar to each other, so are their images". Now we are able to give the representation theorem for perfect fuzzy functions:

Theorem 2 *[3]Let E and F be two M-equivalence relations on X and Y, respectively. Given an ordinary function $f : X \to Y$ extensional w.r.t. E and F, a fuzzy relation $\rho \in L^{X \times Y}$, defined by*

$$\rho(x,y) = F(f(x),y), \; \forall x \in X, \; \forall y \in Y, \qquad (1)$$

is a perfect fuzzy function from X to Y w.r.t. E and F.

Conversely, for a given perfect fuzzy function $\rho \in L^{X \times Y}$ from X to Y w.r.t. E and F, there exists at least one ordinary function $f : X \to Y$ extensional w.r.t. E and F satisfying the equality (1).

Given a perfect fuzzy function ρ from X to Y, an ordinary function $f : X \to Y$ fulfilling the conditions in Theorem 2 is called an ordinary description of ρ, and the set of all ordinary descriptions of ρ is denoted by $ORD(\rho)$. If the M-equivalence relation F on Y is particularly taken as an M-equality on Y, then a perfect fuzzy function ρ from X to Y w.r.t. E and F has a unique ordinary description.

Definition 3 *[4]Given M-equivalence relations $E_{\mathbb{R}^2}$ on $\mathbb{R}^2$ and $E_{\mathbb{R}}$ on $\mathbb{R}$, a perfect fuzzy function $\tilde{+}$ ($\tilde{.}$) from $\mathbb{R}^2$ to $\mathbb{R}$ w.r.t. $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$ is called a perfect M-vague addition (multiplication) operation on $\mathbb{R}$ w.r.t. $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$ iff $+ \in ORD(\tilde{+})$ ($. \in ORD(\tilde{.})$), or equivalently*

$$\tilde{+}(x,y,x+y) = \mathbf{1} \; (\tilde{.}(x,y,x.y) = \mathbf{1}), \; \forall x,y \in \mathbb{R}.$$

By virtue of Theorem 2, a perfect M-vague addition (multiplication) operation $\tilde{+}$ ($\tilde{\cdot}$) on $\mathbb{R}$ w.r.t. $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$ can be explicitly given as

$$\tilde{+}(x,y,z) = E_{\mathbb{R}}(x+y,z)\ (\tilde{\cdot}(x,y,z) = E_{\mathbb{R}}(x.y,z)). \quad (2)$$

For the sake of simplicity, we assume in the remaining part of this study that $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$ stand for two fixed M-equivalence relations on $\mathbb{R}^2$ and $\mathbb{R}$ respectively, and use the notions defined in terms of $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$, for example; the perfect vague addition (multiplication) operations and vague products (sums) of a finite number of real numbers, without indicating $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$.

3 THE VAGUE SUM AND PRODUCT OF A FINITE NUMBER OF REAL NUMBERS

In this section, we start with the definition of the vague sum (product) of a finite number of real numbers in vague arithmetic.

Definition 4 *Let $\tilde{+}$ ($\tilde{\cdot}$) be a perfect M-vague addition (multiplication) operation on $\mathbb{R}$. For $n \in \mathbb{N}^+$ with $n \geq 2$ and for $a_1, a_2, ..., a_n \in \mathbb{R}$, the L-fuzzy set $\widetilde{\Sigma}(a_1,a_2,...,a_n)$ ($\widetilde{\Pi}(a_1,a_2,...,a_n)$): $\mathbb{R} \to L$, defined by*

$$\begin{aligned}
[\widetilde{\Sigma}(a_1,a_2,...,a_n)](x) &= \tilde{+}(\sum_{i=1}^{n-1} a_i, a_n, x) \\
([\widetilde{\Pi}(a_1,a_2,...,a_n)](x) &= \tilde{\cdot}(\prod_{i=1}^{n-1} a_i, a_n, x)),
\end{aligned}$$

is called the vague sum (product) of a_1, a_2,..., a_n. For $n=1$, the L-fuzzy sets $\widetilde{\Sigma}(a_1)$ and $\widetilde{\Pi}(a_1)$ of $\mathbb{R}$ are defined by

$$[\widetilde{\Sigma}(a_1)](x) = [\widetilde{\Pi}(a_1)](x) = \left\{ \begin{array}{ll} \mathbf{1} & , \mathit{if}\ x = a_1 \\ \mathbf{0} & , \mathit{if}\ x \neq a_1 \end{array} \right\}.$$

Due to the equality (2), the vague sum (product) $\widetilde{\Sigma}(a_1,a_2,...,a_n)$ ($\widetilde{\Pi}(a_1,a_2,...,a_n)$) of a_1, a_2,..., a_n for $n \geq 2$ is simply calculated as

$$\begin{aligned}
[\widetilde{\Sigma}(a_1,a_2,...,a_n)](x) &= E_{\mathbb{R}}(\sum_{i=1}^{n} a_i, x) \\
([\widetilde{\Pi}(a_1,a_2,...,a_n)](x) &= E_{\mathbb{R}}(\prod_{i=1}^{n} a_i, x)). \quad (3)
\end{aligned}$$

Furthermore, $\widetilde{\Sigma}(a_1,a_2,...,a_n)$ ($\widetilde{\Pi}(a_1,a_2,...,a_n)$) is independent from the choice of the order of the real numbers a_1, a_2,..., a_n. In other words, for any permutation k_1, k_2,..., k_n of $\{1,2,...,n\}$, we have

$$\begin{aligned}
\widetilde{\Sigma}(a_1,a_2,...,a_n) &= \widetilde{\Sigma}(a_{k_1},a_{k_2},...,a_{k_n}) \\
(\widetilde{\Pi}(a_1,a_2,...,a_n) &= \widetilde{\Pi}(a_{k_1},a_{k_2},...,a_{k_n})).
\end{aligned}$$

The representation of the notions of the vague sum and product in the form (3) provides a very useful tool to extend various elementary properties of the sum and product of a finite number of real numbers in the usual arithmetic to vague arithmetic. For m, $n \in \mathbb{N}^+$ and for a_1, a_2,..., $a_{n+m} \in \mathbb{R}$, the equalities

$$(\sum_{i=1}^{n} a_i) + (\sum_{i=1}^{m} a_{n+i}) = \sum_{i=1}^{n+m} a_i \text{ and } (\prod_{i=1}^{n} a_i).(\prod_{i=1}^{m} a_{n+i}) = \prod_{i=1}^{n+m} a_i$$

are two such well-known elementary properties. As an example, we show in the following theorem how these properties can be carried to vague arithmetic.

Theorem 5 *Let $\tilde{+}$ ($\tilde{\cdot}$) be a perfect M-vague addition (multiplication) operation on $\mathbb{R}$. If $E_{\mathbb{R}}$ is regular w.r.t. "$+$" ("."), then for m, $n \in \mathbb{N}^+$ and for $a_1, a_2, ..., a_{n+m}, x, y, z \in \mathbb{R}$,*

$$\begin{aligned}
& [\widetilde{\Sigma}(a_1,a_2,...,a_n)](x) * [\widetilde{\Sigma}(a_{n+1},a_{n+2},...,a_{n+m})](y) \\
& *[\widetilde{\Sigma}(a_1,a_2,...,a_{n+m})](z) \\
\leq\ & E(x+y,z) \\
& ([\widetilde{\Pi}(a_1,a_2,...,a_n)](x) * [\widetilde{\Pi}(a_{n+1},a_{n+2},...,a_{n+m})](y) \\
& *[\widetilde{\Pi}(a_1,a_2,...,a_{n+m})](z) \\
\leq\ & E(x.y,z))
\end{aligned}$$

In an analogous manner to the integral multiples (powers) of real numbers in usual arithmetic, vague integral multiples (powers) of real numbers in vague arithmetic can be defined as a particular case of the notion of the vague sum (product) of a finite number of real numbers:

Definition 6 *Let $\tilde{+}$ ($\tilde{\cdot}$) stand for a perfect M-vague addition (multiplication) operation on $\mathbb{R}$.*

(i) For $n \in \mathbb{N}^+$, $a \in \mathbb{R}$ and $a_1 = a_2 = ... = a_n$, the L-fuzzy set $\widetilde{na}$ ($\widetilde{a^n}$): $\mathbb{R} \to L$, defined by $\widetilde{na} = \widetilde{\Sigma}(a_1,a_2,...,a_n)$ ($\widetilde{a^n} = \widetilde{\Pi}(a_1,a_2,...,a_n)$), is called the vague n-th multiple (the vague n-th power) of a.

(ii) For $n \in \mathbb{N}^+$ and $a \in \mathbb{R}$ ($a \in \mathbb{R} - \{0\}$), the L-fuzzy set $\widetilde{-na}$ ($\widetilde{a^{-n}}$): $\mathbb{R} \to L$, defined by $\widetilde{-na} = \widetilde{n(-a)}$ ($\widetilde{a^{-n}} = \widetilde{(\frac{1}{a})^n}$), is called the vague $(-n)$-th multiple (the vague $(-n)$-th power) of a.

(iii) For $n = 0$ and $a \in \mathbb{R}$ ($a \in \mathbb{R} - \{0\}$), the L-fuzzy set $\widetilde{0a}$ ($\widetilde{a^0}$): $\mathbb{R} \to L$, defined by

$$[\widetilde{0a}](x) = \left\{ \begin{array}{ll} \mathbf{1} & , \text{if } x = 0 \\ \mathbf{0} & , \text{if } x \neq 0 \end{array} \right\} \left([\widetilde{a^0}](x) = \left\{ \begin{array}{ll} \mathbf{1} & , \text{if } x = 1 \\ \mathbf{0} & , \text{if } x \neq 1 \end{array} \right\} \right),$$

is called the vague 0-th multiple (the vague 0-th power) of a.

In a similar fashion to the vague sum (product) of a finite number of real numbers, for $n \in \mathbb{Z}$ with $|n| \geq 2$, the vague n-th multiple $\widetilde{na}$ (power $\widetilde{a^n}$) of $a \in \mathbb{R}$ can be simply given as

$$[\widetilde{na}](x) = E_{\mathbb{R}}(n.a, x) \; ([\widetilde{a^n}](x) = E_{\mathbb{R}}(a^n, x)), \; \forall x \in \mathbb{R}. \tag{4}$$

Referring to the equality (4), various elementary properties of the integral multiples (powers) of real numbers can be easily carried to the vague integral multiples (powers) of real numbers in vague arithmetic. In order to show how this can be done, for m, $n \in \mathbb{Z}$ and for $a \in \mathbb{R}$ ($a \in \mathbb{R} - \{0\}$), we now extend the well-known relations

$$\begin{array}{rcl} (n.a) + (m.a) & = & (n+m).a \text{ and } n.(m.a) = (n.m).a \\ ((a^n).(a^m) & = & a^{n+m} \text{ and } (a^m)^n = a^{m.n}) \end{array}$$

in usual arithmetic to vague arithmetic as follows:

Theorem 7 *Let $\tilde{+}$ ($\tilde{\cdot}$) be a perfect M-vague addition (multiplication) operation on $\mathbb{R}$, and $E_{\mathbb{R}}$ a regular M-equivalence relation w.r.t. "+" ("."). Then for m, $n \in \mathbb{Z}$ and for $a \in \mathbb{R}$ ($a \in \mathbb{R} - \{0\}$), the following relations are valid for all $x, y, z \in \mathbb{R}$:*

$$\begin{array}{rcl} [\widetilde{na}](x) * [\widetilde{ma}](y) * [\widetilde{(n+m)a}](z) & \leq & E(x+y, z) \\ \text{and } [\widetilde{(m.n)a}](x) * ([\widetilde{ma}](z))^n * [\widetilde{nz}](y) & \leq & E(x, y) \\ ([\widetilde{a^n}](x) * [\widetilde{a^m}](y) * [\widetilde{a^{n+m}}](z) & \leq & E(x.y, z) \\ \text{and } [\widetilde{a^{m.n}}](x) * ([\widetilde{a^m}](z))^n * [\widetilde{z^n}](y) & \leq & E(x, y)). \end{array}$$

4 PRACTICAL APPLICATIONS AND EXAMPLES

Indistinguishability operators [9, 11] (i.e. M-equivalence relations for the particular po-monoid $M = ([0,1], \leq, *)$) are natural mathematical tools to represent the uncertainty resulting from the discrete scales of measurement instruments [6, 9]. Under the consideration of this uncertainty and its mathematical representation-indistinguishability operators, the notion of the vague sum (product) of a finite number of real numbers become a mathematical modelling of the sum (product) of a finite number of possible outcomes of a quantity in a measurement process. The following example is designed to demonstrate how the notions of the vague sum and product can be practically used in a measurement process.

Example 8 *(i) In the measurement of the perimeter of a polygon P with n different sides (or simply an n-gon), let us consider a rod having centimeter (cm) readings as our measurement instrument. Because of the discrete readings of this rod, it is natural to model the indistinguishability of any two readings x and y on this rod gradually. For a suitably chosen indistinguishability operator $E_{\mathbb{R}}$ on $\mathbb{R}$, the indistinguishability of x and y can be graded as the real number $E_{\mathbb{R}}(x,y)$. In this example, for the Lukasiewicz's t-norm Lck, i.e. $Lck(\alpha, \beta) = max\{\alpha + \beta - 1, 0\}$, $\forall \alpha, \beta \in [0,1]$, we take $E_{\mathbb{R}}$ as the Lck-indistinguishability operator:*

$$E_{\mathbb{R}}(x,y) = 1 - min\{|x-y|, 1\}, \; \forall x, y \in \mathbb{R} \; [9].$$

Furthermore, if we consider the Lck-indistinguishability operator $E_{\mathbb{R}^2}$ defined by

$$E_{\mathbb{R}^2}((x,y),(x',y')) = E_{\mathbb{R}}(x+y, x'+y'), \; \forall x, x', y, y' \in \mathbb{R},$$

then since "+" is extensional w.r.t. $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$, and by virtue of Theorem 2, the fuzzy relation $\tilde{+}$ given by (2) is a perfect M-vague addition operation on $\mathbb{R}$ w.r.t. $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$. Therefore, if we measure the sides of lengths of P as a_1, a_2, ..., a_n, then the perimeter of the n-gon P in the unit of cm will be the vague sum $\tilde{\Sigma}(a_1, a_2, ..., a_n)$ of a_1, a_2, ..., a_n w.r.t. $\tilde{+}$. Here $\tilde{\Sigma}(a_1, a_2, ..., a_n)$ is explicitly given as

$$\begin{array}{rcl} [\tilde{\Sigma}(a_1, a_2, ..., a_n)](x) & = & E_{\mathbb{R}}(\sum\limits_{i=1}^{n} a_i, x) \\ & = & 1 - min\{\left| \sum\limits_{i=1}^{n} a_i - x \right|, 1\}, \end{array}$$

for all $x \in \mathbb{R}$, and defines a triangular fuzzy number centered at $\sum\limits_{i=1}^{n} a_i$ (see Figure 1).

If P is particularly chosen as a regular polygon with the measured sides of length a, then the perimeter of P now turns to the vague n-th multiple $\widetilde{na}$ of a w.r.t. $\tilde{+}$, and is simply computed as

$$[\widetilde{na}](x) = E_{\mathbb{R}}(n.a, x) = 1 - min\{|n.a - x|, 1\}, \; \forall x \in \mathbb{R}.$$

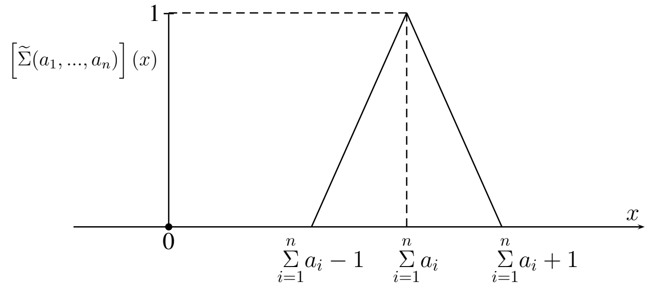

Figure 1:

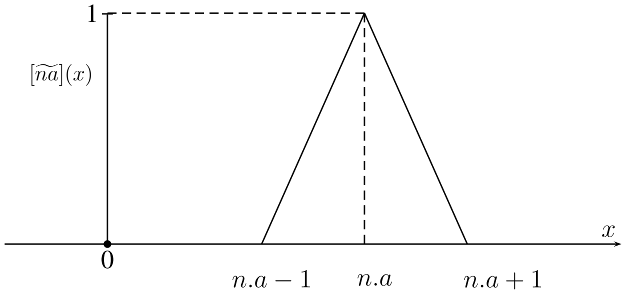

Figure 2:

Here $\widetilde{na}$ is a triangular fuzzy number centered at $n.a$, and is represented in Figure 2.

(ii) In order to supply a practical example for the notions of the vague product and the vague integral power, let us deal with the measurement of volume of a right rectangular prism U, and consider the same rod in (i). For the gradual representation of the indistinguishability of the readings of the rod, we now operate with the Pr-indistinguishability operator $E_{\mathbb{R}}$:

$$E_{\mathbb{R}}(x,y) = \left\{ \begin{array}{ll} min\left\{\frac{x}{y},\frac{y}{x}\right\} & , if\ x.y > 0 \\ 0 & , if\ x.y < 0 \\ 1 & , if\ x = y = 0 \end{array} \right\}$$

for all $x,y \in \mathbb{R}$, where Pr stands for the product t-norm, i.e. $Pr(\alpha,\beta) = \alpha.\beta$, $\forall \alpha,\beta \in [0,1]$. Taking into consideration the Pr-indistinguishability operator $E_{\mathbb{R}^2}$ on $\mathbb{R}^2$, defined by

$$E_{\mathbb{R}^2}((x,y),(x',y')) = E_{\mathbb{R}}(x.y,x'.y'),\ \forall x,y,x',y' \in \mathbb{R},$$

it is easy to observe that the fuzzy relation $\tilde{\cdot}$ given by (2) is a perfect M-vague multiplication operation on $\mathbb{R}$ w.r.t. $E_{\mathbb{R}^2}$ and $E_{\mathbb{R}}$. Assume that a_1, a_2 and a_3 are the measured values of the sides of lengths of U. Then the volume of U in the unit of cm^3 will be the vague product $\widetilde{\Pi}(a_1,a_2,a_3)$ of a_1, a_2 and a_3 w.r.t. $\tilde{\cdot}$. $\widetilde{\Pi}(a_1,a_2,a_3)$ is explicitly written as

$$[\widetilde{\Pi}(a_1,a_2,a_3)](x) = E_{\mathbb{R}}(a_1.a_2.a_3,x)$$

$$= \left\{ \begin{array}{ll} \frac{a_1.a_2.a_3}{x} & , if\ x > a_1.a_2.a_3 \\ \frac{x}{a_1.a_2.a_3} & , if\ 0 < x \leq a_1.a_2.a_3 \\ 0 & , if\ x \leq 0 \end{array} \right\}$$

for all $x \in \mathbb{R}$, and is sketched in Figure 3.

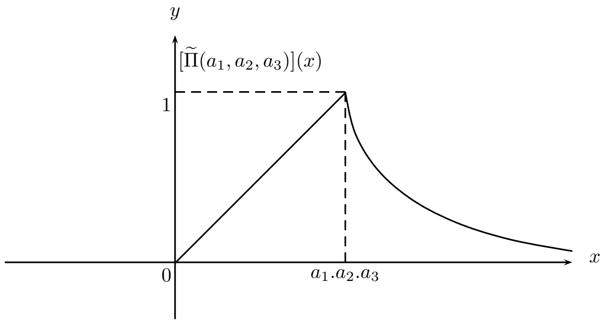

Figure 3:

In a similar fashion to (i), if U forms a cube, and if we measure one of the sides of length of U as a, then the volume of U is calculated as the vague third power $\widetilde{a^3}$ of a. $\widetilde{a^3}$ is easily calculated as

$$[\widetilde{a^3}](x) = E_{\mathbb{R}}(a^3,x) = \left\{ \begin{array}{ll} \frac{a^3}{x} & , if\ x > a^3 \\ \frac{x}{a^3} & , if\ 0 < x \leq a^3 \\ 0 & , if\ x \leq 0 \end{array} \right\}$$

for all $x \in \mathbb{R}$, and is illustrated in Figure 4.

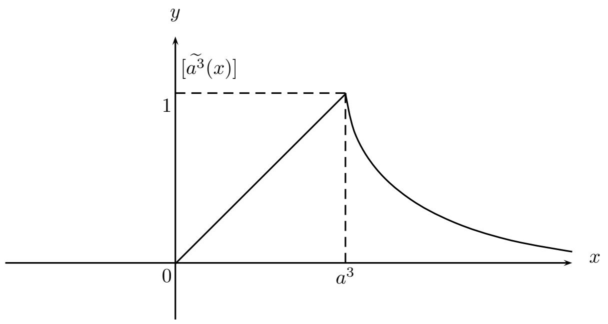

Figure 4:

As a concluding remark, it should be noted that the *Lck* (*Pr*)-indistinguishability operator $E_{\mathbb{R}}$ in Example 8 also provides an example for a regular *M*-equivalence relation w.r.t. “$+$” (“$.$”) as required in Theorem 5 and Theorem 7.

5 CONCLUSIONS AND FUTURE STUDIES

In this paper, the notions of the vague sum (product) of a finite number of real numbers and the vague integral multiples (powers) of real numbers, which are nothing but a special case of the notions of the vague product of a finite number of elements and the vague integral powers of elements in vague groups presented in [7, 8], have been introduced by means of a perfect *M*-vague addition (multiplication) operation on $\mathbb{R}$. Thereafter, without needing any vague algebraic notion, it is shown that some elementary properties of these notions can be easily deduced from their representation properties and some elementary algebraic properties of the usual addition (multiplication) operation. Furthermore, the present paper reveals that whenever the uncertainty occurring due to the indistinguishability of the possible outcomes in a measurement process is the matter of subject, and is represented by indistinguishability operators (more generally, *M*-equivalence relations), the vague sum (product) of a finite number of real numbers provides a more realistic and informative mathematical tool to compute the sum (product) of a finite number of possible values of a quantity in a measurement process.

The formulation of the order-theoretic structure of the real line equipped with a perfect *M*-vague addition $\tilde{+}$ operation and a perfect *M*-vague multiplication operation $\tilde{\cdot}$, which is denoted by $(\mathbb{R}, \tilde{+}, \tilde{\cdot})$ and called a vague real line, lies at the heart of the development of an axiomatic and a sound theory of a vague real line. The solution to this problem is left to future studies.

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