Indistinguishability in Cooperative Games.

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Abstract

Game Theory has proven to be a useful tool for modelling decision problems in competitive and collaborative environments. Cooperative games, in turn, promote the bargaining and the formation of coalitions, and deals with problems as the redistribution of gained payoff between members of coalitions. This paper introduces the concept of indistinguishability among players, providing constructive methods for computing the T-indistinguishability operator associated to any cooperative game.

Keywords: T-indistinguishability operators, game theory, cooperative games.

1 Introduction.

The significance of classification as a key task underlying most cognitive activities is today widely recognized.

Moreover, classification as the process of grouping or clustering according to a certain criterion of similarity tends to be intimately related to traditional notions of identity, indiscernibility and indistinguishability. All these concepts have a long tradition as subjects of discussion within a great number of fields, ranging from philosophy and psychology to mathematics.

The standard way of approaching the concept of identity is linked to a tradition that can be traced back at least to Leibniz, whose Law of Identity is usually written in a second-order language as

$$x \approx y \Leftrightarrow \forall P : P(x) \Leftrightarrow P(y) \tag{1}$$

where x and y denote individuals and P ranges over the set of properties.

Leibniz's Law, which is a conjunction of the principles of the Identity of Indiscernible and Indiscernibility of Identical, is intended to express the concept of identity as agreement with respect to all properties. The original postulate may have evolved towards more elaborated formulations but the main idea behind remains the same.

When all the properties involved are entirely precise (lack of uncertainty), what we obtain is the classical equality where two individuals are considered equal if and only if they share the same set of properties. What happens, however, when imprecision arises, as in the case of properties which are fulfilled only up to a degree? Thus, because certain individuals will be more similar than others, the need for a gradual notion of equality arises.

Equality satisfies transitivity only in the context of pure mathematics [3]. In the real world, "equal" really means "indistinguishable". Indeed, since a chain of objects that are usually indistinguishable can lead from one which clearly seems to be compatible with a given property to one which clearly does not, sorites paradox and the corresponding break in transitivity ensues.

All these considerations show that certain contexts that are pervaded with uncertainty ask for a more flexible concept of equality that goes beyond the rigidity of the classic concept of equality.

T-indistinguishability operator [1] seems to be a good candidate for the more general version of the concept of equality we are searching for.

The relative nature of the notion of equality can also be inferred from the formalization of Leibniz's Law (1). Indeed, since every context or theory defines its own set of descriptive attributes (set of properties in the aforementioned formalization), the direct application of Leibniz's Law yields different equality criteria depending on the context of discourse; therefore, every theory handles its own notion of equality over the objects in question.

In this paper, we aim to study the concept of indistinguishability within the framework of Cooperative Game Theory.

The paper is organized as follows: section 2 provides basic definitions on T-indistinguishability operators and Cooperative Games. The main result of the paper is given in Section 3 by presenting a constructive method for computing the indistinguishability degrees between players in cooperative games. Finally, an example is given in section 4.

2 Definitions.

This section provides several definitions and propositions that will be used throughout the paper.

2.1 On T-indistinguishability operators.

Definition 1 *Given a continuous t-norm T, its residuation $\hat{T}$ is defined* $\forall x, y \in [0,1]$*:*

$$\hat{T}(x|y) = sup\{\alpha \in [0,1] : T(\alpha, x) \leqslant y\}. \quad (2)$$

Let us recall the residuations for the three most commonly used t-norms.

1. When $T(x,y) = \min(x,y)$ then
$$\hat{T}(x|y) = \begin{cases} 1 & x \leqslant y \\ y & \text{otherwise.} \end{cases} \quad (3)$$
2. When $T(x,y) = x \cdot y$ then
$$\hat{T}(x|y) = \min(1, \frac{y}{x}). \quad (4)$$
3. When $T(x,y) = \max(x+y-1, 0)$ then
$$\hat{T}(x|y) = \min(1-x+y, 1). \quad (5)$$

Definition 2 *Given a continuous t-norm T, its biresiduation $\overleftrightarrow{T}$ is defined as:*

$$\overleftrightarrow{T}(x,y) = \min(\hat{T}(x|y), \hat{T}(y|x)). \quad (6)$$

Definition 3 *A fuzzy relation E on a set X is a T-indistinguishability operator if and only for all x, y, z of X satisfies the following properties*

$E(x,x) = 1$ *(reflexivity)*
$E(x,y) = E(y,x)$ *(symmetry)*
$E(x,z) \geqslant T(E(x,y), E(y,z))$ *(T-transitivity).*

One problem commonly faced when studying such relations is how to effectively build them.

The traditional approach relies on computing the transitive closure from a reflexive and symmetric relation. This method, however, has not proved to be fully satisfactory because of the computational cost involved (the closure is computed as the supremum of *max-T* powers of the original relation) and primarily because of the distortion suffered by the initial values.

These weaknesses were surmounted by the introduction of the representation theorem for T-indistinguishability operators .

Theorem 1 *[4] Let E be a map from $X \times X$ into $[0,1]$ and T be a continuous t-norm . E is a T-indistinguishability operator if and only if a family $\{h_j\}_{j \in J}$ of fuzzy sets exists in X, such that*

$$E(x,y) = \inf_{j \in J} \hat{T}\big(\max(h_j(x), h_j(y)) \mid \min(h_j(x), h_j(y))\big) \quad (7)$$

2.2 On Cooperative Game theory.

The notion of game was introduced as a mathematical abstraction for modelling decision problems in competitive and collaborative situations where each participant has only partial influence over the set of variables governing the final outcome.

The publication of *"Game Theory and Economic Behavior"* by Von Neumann and Morgensten [2] represented the formal inception of Game Theory. Since then it has evolved considerably, allowing its application to a wide set of areas such as economics, politics, psychology, ...

Games are usually divided in cooperative and non-cooperative games. Non-cooperative games deals with situations in which players select their optimal strategy based on their guesses about which strategies their opponents are more likely to choose. By contrast, cooperative games promote the bargaining and the formation of coalitions in order to increase the amount of "gain" to be redistributed among its components.

This paper is centered in cooperative games. Let us now recall some basic definitions.

Definition 4 *Let N, with $|N| \geq 2$, be the set of players. Any subset $S \subseteq N$ is called a coalition of players. Sets $\emptyset$ and N are called the empty coalition and the grand coalition, respectively.*

Definition 5 *A cooperative game is given by the pair (N, v) where N is the set of players and v is the characteristic function of the game given by*

$$v : 2^N \longrightarrow \Re$$

where $v(S)$ is interpreted as the value (i.e worth or power) of coalition S when its members act together as a unit.

Definition 6 *A game (N, v) is superadditive iff for all coalitions $S, T \subseteq N$ such that $S \cap T = \emptyset$ it holds*

$$v(S \cup T) \geq v(S) + v(T) \tag{8}$$

Definition 7 *A game (N, v) is monotone iff for all coalitions $S, T \subseteq N$:*

$$S \subseteq T \Longrightarrow v(S) \leq v(T) \tag{9}$$

Definition 8 *A game (N, v) is simple iff for every coalition $S, T \subseteq N$, either $v(S) = 0$ or $v(S) = 1$.*

In a simple game a coalition S is called a winning coalition if $v(S) = 1$ and a losing coalition in other case.

Typical examples of simple games are:

- Majority rule game, where $v(S) = 1$ iff $|S| > \frac{n}{2}$, and $v(S) = 0$ otherwise.
- Unanimity game, where $v(S) = 1$ iff $S = N$, and $v(S) = 0$ otherwise.
- Dictator game, where given a distinguished player a, $v(S) = 1$ iff $a \in S$, and $v(S) = 0$ otherwise.

Definition 9 *A game (N, v) is symmetric iff for all coalition S, $v(S)$ depends only on the number of elements of S, say $v(S) = f(|S|)$ for some function f.*

Definition 10 *Given a game $G = (N, v)$, we define its associated normal game G' as the game (N, v') with characteristic function v' defined as*

$$\forall S \subseteq N : v'(S) = \frac{v(S)}{\max_{T \subseteq N} v(T)}$$

3 Players indistinguishability.

In this section we present a theorem in order to define and compute the indistinguishability degree among players of any cooperative game.

First, let us introduce a lemma which is necessary for proving the main theorem.

Lemma 1 *[4] For all continuous t-norm T and $\forall x, y, z \in X$ it holds that*

$$\overset{\leftrightarrow}{T}(x, z) \geqslant T\big(\overset{\leftrightarrow}{T}(x, y), \overset{\leftrightarrow}{T}(y, z)\big). \tag{10}$$

Theorem 2 *Let $G = (N, v)$ a normal cooperative game. Then for all players $a, b \in N$, the relation*

$$E(a, b) = \min_{A \in \wp(N - \{a,b\})} \overset{\leftrightarrow}{T}\big(v(\{a\} \cup A), v(\{b\} \cup A)\big) \tag{11}$$

is a T-indistinguishability operator .

Proof 1 *Let us prove that E is reflexive, symmetric and T-transitive.*

Reflexivity: *For all player $a \in N$ it holds that*

$$E(a,a) = \min_{A\in\wp(N-\{a\})} \overset{\leftrightarrow}{T}\big(v(\{a\}\cup A), v(\{a\}\cup A)\big) = 1.$$

Symmetry: *Immediate from the symmetry of the operator* $\overset{\leftrightarrow}{T}$ *.*

T-transitivity: *by proving that for all players* $a,b,c \in N$ *and* $\forall Z \in \wp(N-\{a,c\})$*, it holds that*

$$\overset{\leftrightarrow}{T}\big(v(\{a\}\cup Z), v(\{c\}\cup Z)\big) \geqslant T(E(a,b),E(b,c)) \quad (12)$$

then

$$E(a,c) = \min_{Z\in\wp(N-\{a,c\})} \overset{\leftrightarrow}{T}\big(v(\{a\}\cup Z), v(\{c\}\cup Z)\big) \geqslant T(E(a,b),E(b,c))$$

and T-transitivity would be proved.

Let us show that the above inequality (12) is indeed satisfied. Let $Z \in \wp(N-\{a,c\})$*. Then, we can consider two cases:*

1. *Player* $b \notin Z$*. Then*

$$Z \in (\wp(N-\{a,c\} \cap \wp(N-\{b\})) = \wp(N-\{a,b,c\})$$

and by lemma (1) it holds that

$$\overset{\leftrightarrow}{T}\big(v(\{a\}\cup Z), v(\{c\}\cup Z)\big) \geqslant \geqslant T\Big(\overset{\leftrightarrow}{T}\big(v(\{a\}\cup Z), v(\{b\}\cup Z)\big), \overset{\leftrightarrow}{T}\big(v(\{b\}\cup Z), v(\{c\}\cup Z)\big)\Big) \quad (13)$$

and since

$$\wp(N-\{a,b,c\}) \subset \wp(N-\{a,b\})$$
$$\wp(N-\{a,b,c\}) \subset \wp(N-\{b,c\})$$

we have

$$\geqslant T\Big(\min_{V\in\wp(N-\{a,b,c\})} \overset{\leftrightarrow}{T}\big(v(\{a\}\cup V), v(\{b\}\cup V)\big), \min_{W\in\wp(N-\{a,b,c\})} \overset{\leftrightarrow}{T}\big(v(\{b\}\cup W), v(\{c\}\cup W)\big)\Big)$$
$$\geqslant T\Big(\min_{U\in\wp(N-\{a,b\})} \overset{\leftrightarrow}{T}\big(v(\{a\}\cup U), v(\{b\}\cup U)\big), \min_{Y\in\wp(N-\{b,c\})} \overset{\leftrightarrow}{T}\big(v(\{b\}\cup Y), v(\{c\}\cup Y)\big)\Big) = T(E(a,b),E(b,c)).$$

2. *Player* $b \in Z$*. Then* $Z \in \wp(N-\{a,c\})$ *and besides* $Z \notin \wp(N-\{a,b,c\})$ *which entails*

$$\{a\}\cup(Z-\{b\}) \in \wp(N-\{b,c\})$$
$$\{c\}\cup(Z-\{b\}) \in \wp(N-\{a,b\})$$

By lemma (1) it holds that

$$\overset{\leftrightarrow}{T}\big(v(\{a\}\cup Z), v(\{c\}\cup Z)\big) \geqslant T\Big(\overset{\leftrightarrow}{T}\big(v((\{a\}\cup(Z-\{b\}))\cup\{b\}), v((\{a\}\cup(Z-\{b\}))\cup\{c\})\big), \overset{\leftrightarrow}{T}\big(v((\{c\}\cup(Z-\{b\}))\cup\{a\}), v((\{c\}\cup(Z-\{b\}))\cup\{b\})\big)\Big)$$

since trivially

$$\{a\}\cup Z = \{a\}\cup(Z-\{b\})\cup\{b\}$$
$$\{c\}\cup Z = \{c\}\cup(Z-\{b\})\cup\{b\}$$
$$\{a\}\cup(Z-\{b\})\cup\{c\} = \{c\}\cup(Z-\{b\})\cup\{a\}$$

Moreover, as

$$(\{a\}\cup(Z-\{b\})) \in \wp(N-\{b,c\})$$

and

$$(\{c\}\cup(Z-\{b\})) \in \wp(N-\{a,b\})$$

the expression above holds

$$\geqslant T\Big(\min_{U\in\wp(N-\{b,c\})} \overset{\leftrightarrow}{T}\big(v(U\cup\{b\}), v(U\cup\{c\})\big),$$
$$\min_{Y\in\wp(N-\{a,b\})} \overset{\leftrightarrow}{T}\big(v(Y\cup\{a\}), v(Y\cup\{b\})\big)\Big)$$
$$= T(E(b,c), E(a,b))$$

(QED)

The theorem above defines the indistinguishability degree for any pair of players a, b as the minimum of the biresiduation between the values of their respective coalitions when both are accompanied by the same set of players.

As an example, let $G = (\{a,b,c,d\}, v)$ an instance of the dictator game defined in section 2.2, where $v(\{a\}) = v(\{a,b\}) = v(\{a,c\}) = v(\{a,d\}) = v(\{a,b,c\}) = v(\{a,b,d\}) = v(\{a,c,d\}) = v(\{a,b,c,d\}) = 1$ and $v(\{b\}) = v(\{c\}) = v(\{d\}) = v(\{b,c\}) = v(\{b,d\}) = v(\{c,d\}) = v(\{b,c,d\}) = 0$. Then, the T-indistinguishability operator E_2 associated to G is

	a	**b**	**c**	**d**
a	1	0	0	0
b	0	1	1	1
c	0	1	1	1
d	0	1	1	1

which clearly partitions the set of players in two disjoint classes, namely the dictator class $(\{a\})$ and the rest of players $(\{b,c,d\})$.

For symmetric games (see section 2.2), given that the value of coalitions do not depend on the particular members composing them but on the cardinal, the following proposition holds:

Lemma 2 *Let Ω be the set of all T-indistinguishability operators on a domain X. Then Ω is a poset where $\forall E, E' \in \Omega$:*

$$E \leq E' \Leftrightarrow \forall x, y \in X : E(x,y) \leq E'(x,y).$$

The maximal and minimal elements of the poset are, respectively, relation $E^{\bar{1}}$ defined by

$$\forall x, y \in X : E^{\bar{1}}(x,y) = 1$$

and relation $E^{\bar{0}}$ where

$$\forall x, y \in X : E^{\bar{0}}(x,y) > 0 \Leftrightarrow x = y.$$

Proposition 1 *Let $G = (N, v)$ be a symmetric game. Then its associated operator E_2 equals $E^{\bar{1}}$.*

As a trivial corollary of this proposition, operator E_2 associated to majority and unanimity games (see section 2.2) equals indistinguishability operator $E^{\bar{1}}$.

4 An example: Shortest Path Game.

In this section we will illustrate the presented definitions with the introduction of the shortest path games for which some traffic is supposed to be routed through a network where each link is owned by a player who incurs some cost while transporting traffic along his link.

The value of a coalition is the payoff paid by sender player (labelled as s) when the traffic can be transported (which we fix in 100 units) minus the cost derived from using the links. This cost is supposed to be the minimum, thereby maximizing the net payoff so that if some coalition "opens" two possible paths for routing the traffic, only the cheapest path would be consider when computing the net payoff (i.e value) received by the coalition. Obviously, for coalitions in which sender player is not included, their value is zero since without no sender, no traffic and hence no payoff.

Therefore, the following network

```
          a=10        b=20        c=30
        --------> * --------> * ---------
        |                               |
  s ---|                               |-->
        |                               |
        --------> * --------> * ---------
          d=20        e=20        f=50
```

defines the game $G = (\{s,a,b,c,d,e,f\}, v)$ where $\forall S$:

$$v(S) = \begin{cases} 40 & \text{if } \{s, a, b, c\} \in S. \\ 10 & \text{if } \{s, d, e, f\} \in S \text{ and } \{a, b, c\} \notin S. \\ 0 & \text{otherwise.} \end{cases}$$

For game G and taking the Lukasiewicz t-norm, its associated operator E_2 is

	a	**b**	**c**	**d**	**e**	**f**	**s**
a	1	1	1	0	0	0	0.75
b	1	1	1	0	0	0	0.75
c	1	1	1	0	0	0	0.75
d	0	0	0	1	1	1	0
e	0	0	0	1	1	1	0
f	0	0	0	1	1	1	0
s	0.75	0.75	0.75	0	0	0	1

Note that two classes has naturally formed, namely $\{a, b, c\}$ and $\{d, e, f\}$, corresponding to the two possible paths connecting origin and destination.

Moreover, sender player s has greater indistinguishability degree with players owning links belonging to the solution path ($s \longrightarrow a \longrightarrow b \longrightarrow c$) than with the rest of players.

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References

[1] J. Jacas, and J. Recasens, "Fuzzy T-transitive Relations: Eigenvectors and Generators", Fuzzy Sets and Systems, number 72, pp. 147-154, 1995.

[2] J. von Neumann, and O. Morgenstern, "Theory of Games and Economic Behavior", Princeton University Press , Princeton, New Jersey, 1944.

[3] H. Poincaré, La science et l'hypothése, Flammarion, Paris, 1902.

[4] L. Valverde, "On the structure of F-indistinguishability operators" , Fuzzy Sets and Systems, number 17, pp. 313-328, 1985.

[5] L. A. Zadeh, "Similarity Relations and Fuzzy Orderings", Information Science,number 3 , pp. 177-200. 1971.